333 Exam 1 Solutions.

1. $\dfrac{dy}{dx} = \dfrac{x-e^{-x}}{y+e^{y}} \Rightarrow (y+e^{y})\,dy = (x-e^{-x})\,dx$

$$\Rightarrow \boxed{\frac{1}{2}y^2+e^{y} = \frac{1}{2}x^2+e^{-x}+C}$$

2. $\dfrac{dy}{dx} = -\dfrac{4x+3y}{2x+y} = -\dfrac{4+3\frac{y}{x}}{2+\frac{y}{x}}$

let $v = \dfrac{y}{x} \Rightarrow y = xv \Rightarrow \dfrac{dy}{dx} = 1\cdot v + x\dfrac{dv}{dx}$

$$\Rightarrow v + x\frac{dv}{dx} = -\frac{4+3v}{2+v}$$

$$x\frac{dv}{dx} = -\frac{4+3v}{2+v} - v = \frac{-4-3v}{2+v} - \frac{v(2+v)}{2+v} = \frac{-4-3v-2v-v^2}{2+v}$$

$$x\frac{dv}{dx} = \frac{-4-5v-v^2}{2+v} = -\frac{(v^2+5v+4)}{2+v} = -\frac{(v+4)(v+1)}{v+2}$$

$\Rightarrow$

$$\Rightarrow x\frac{dv}{dx} = \frac{-(v+4)(v+1)}{v+2}$$

$$\Rightarrow \frac{+(v+2)}{(v+1)(v+4)}\,dv = \frac{-1}{x}\,dx$$

$$\Rightarrow \underbrace{\int \frac{v+2}{(v+1)(v+4)}\,dv}_{\text{partial fractions}} = -\ln|x| + C$$

$$\frac{v+2}{(v+1)(v+4)} = \frac{A}{v+1} + \frac{B}{v+4} \Rightarrow v+2 = A(v+4) + B(v+1)$$

$$v=-4 \Rightarrow -2 = -3B \quad B = 2/3$$

$$v=-1 \Rightarrow 1 = 3A \quad A = 1/3$$

$$\int \left[\frac{1/3}{v+1} + \frac{2/3}{v+4}\right] dv = -\ln|x| + C$$

$$\frac{1}{3}\ln|v+1| + \frac{2}{3}\ln|v+4| = -\ln|x| + C$$

$$\frac{1}{3}\left(\ln|v+1| + 2\ln|v+4|\right) = -\ln|x| + C$$

$$\Rightarrow \ln\left(|v+1|\cdot|v+4|^2\right) = -3\ln|x| + C$$

$$\Rightarrow \ln\left(\left|\frac{y}{x}+1\right|\cdot\left|\frac{y}{x}+4\right|^2\right)+3\ln|x| = c$$

$$\Rightarrow \ln\left(\left|\frac{y}{x}+1\right|\cdot\left|\frac{y}{x}+4\right|^2 \cdot |x|^3\right) = c$$

$$\ln\left(|y+x|\cdot|y+4x|^2\right) = c$$

$$\Rightarrow \boxed{|y+x|\cdot|y+4x|^2 = c}$$

3. $\dfrac{dy}{dt} + 3y = e^{-2t} + 4.$

Using the method of integrating factors,

$g(t) = 3\ ,\quad b(t) = e^{-2t} + 4.$

$\int g(t)\,dt = 3t \Rightarrow \mu(t) = e^{3t}$

$$\Rightarrow y(t) = \frac{1}{e^{3t}}\int e^{3t}(e^{-2t}+4)\,dt$$

$$= e^{-3t}\left(\int e^{t} + 4e^{3t}\ dt\right)$$

→

$$\Rightarrow y(t) = e^{-3t}\left(e^{t} + \frac{4}{3}e^{3t} + C\right)$$

$$= e^{-2t} + \frac{4}{3} + Ce^{-3t}$$

$$\Rightarrow \boxed{y(t) = e^{-2t} + Ce^{-3t} + \frac{4}{3}}$$

4. $y'' + \frac{1}{2}y' + \frac{1}{16}y = 0.$

The char. eqn. is $r^2 + \frac{1}{2}r + \frac{1}{16} = 0.$

$r = \dfrac{-\frac{1}{2} \pm \sqrt{\frac{1}{4} - 4\cdot\frac{1}{16}}}{2} = -\frac{1}{4}.$ $\Rightarrow$ The char. eqn. has a repeated real root $r_1 = r_2 = -\frac{1}{4}.$

$$\Rightarrow \boxed{y(t) = k_1 e^{-t/4} + k_2 t e^{-t/4}}$$

$\rightarrow$

5. $y'' + 6y = 0, \quad y(0) = 1, \quad y'(0) = -1$

The char. eqn. is $r^2 + 6 = 0 \Rightarrow r^2 = -6$

$\Rightarrow r = \pm\sqrt{-6} = \pm i\sqrt{6} \Rightarrow \lambda = 0, \mu = \sqrt{6}$

The general solution is

$y(t) = k_1 \cos(\sqrt{6}\,t) + k_2 \sin(\sqrt{6}\,t)$. $\quad y(0) = 1 \Rightarrow k_1 = 1$

$y'(t) = -k_1\sqrt{6}\sin(\sqrt{6}\,t) + k_2\sqrt{6}\cos(\sqrt{6}\,t) \quad y'(0) = -1 \Rightarrow k_2 = -\dfrac{\sqrt{6}}{6}$

$\Rightarrow \boxed{y(t) = \cos(\sqrt{6}\,t) - \dfrac{\sqrt{6}}{6}\sin(\sqrt{6}\,t)}$

6. $ay'' + by' + cy = 0$

$y(t) = e^{-t} + 7e^{-2t}$

The roots of the char. eqn. must be $r_1 = -1, r_2 = -2$.

$(r+1)(r+2) = 0. = r^2 + 3r + 2.$

$\Rightarrow \boxed{y'' + 3y' + 2y = 0}$ is one such DE.

7. $ay'' + by' + cy = 0$

$y(t) = 5 + e^{-3t}$

The roots of the char. eqn. must be $r_1 = 0$, $r_2 = -3$.

$\Rightarrow (r-0)(r+3) = r^2 + 3r = 0$

$\Rightarrow \boxed{y'' + 3y' = 0}$ is one such DE.

8. No such differential equation exists.

9. Bernoulli Equations.

$$\frac{dy}{dt} + p(t)y = q(t)y^n$$

(a) $n = 0 \Rightarrow \frac{dy}{dt} + p(t)y = q(t)$. $\Rightarrow$ use the method of integrating factors to write the solution.

$$\Rightarrow y(t) = \frac{1}{e^{\int p(t)dt}}\left(\int e^{\int p(t)dt} q(t)\, dt\right).$$

(b) $n=1 \Rightarrow \dfrac{dy}{dt} + p(t)y = q(t)y$

$$\Rightarrow \frac{dy}{dt} = (q(t)-p(t))y$$

$$\Rightarrow \frac{1}{y}\,dy = (q(t)-p(t))\,dt$$

$$\Rightarrow \ln|y| = \int (q(t)-p(t))\,dt.$$

(c) $\dfrac{dy}{dt} + p(t)y = q(t)y^n, \quad n \neq 0,1$

Let $v = y^{1-n} \Rightarrow \dfrac{dv}{dt} = (1-n)y^{(1-n)-1}\dfrac{dy}{dt} = (1-n)y^{-n}\dfrac{dy}{dt}$

$$\Rightarrow \frac{dy}{dt} = \frac{1}{1-n}y^n\frac{dv}{dt}$$

$$\Rightarrow \frac{1}{1-n}y^n\frac{dv}{dt} + p(t)y = q(t)y^n$$

Dividing both sides by y^n, we obtain:

$$\frac{1}{1-n}\frac{dv}{dt} + p(t)\,y^{1-n} = q(t)$$

$$\Rightarrow \frac{1}{1-n}\frac{dv}{dt} + p(t)\,v = q(t)$$

$$\Rightarrow \boxed{\frac{dv}{dt} + (1-n)p(t)v = (1-n)q(t)}$$, which is a

linear differential equation for $v(t)$.

(d) $t^2\frac{dy}{dt} + 2ty - y^3 = 0, \ t>0$

$$\Rightarrow \frac{dy}{dt} + \frac{2}{t}y - \frac{y^3}{t^2} = 0$$

$$\Rightarrow \frac{dy}{dt} + \frac{2}{t}y = \frac{1}{t^2}y^3.$$

$\rightarrow$

As suggested in (c), make the substitution

$v = y^{1-3} = y^{-2}$. Then the DE becomes

$$\frac{dv}{dt} + (1-3)\cdot\frac{2}{t}v = (1-3)\frac{1}{t^2}$$

$$\frac{dv}{dt} - \frac{4}{t}v = -2t^{-2}.$$ This is a linear DE, so we can solve it by the method of integrating factors.

$g(t) = -\frac{4}{t}$ $b(t) = -2t^{-2}$

$$\int g(t)\,dt = -4\ln t \Rightarrow \mu(t) = e^{-4\ln t} = e^{\ln t^{-4}} = t^{-4}$$

$$\Rightarrow v(t) = \frac{1}{t^{-4}}\left(\int t^{-4}\cdot -2t^{-2}\,dt\right)$$

$$= t^4\left(-2t^{-6}\,dt\right) = t^4\left(\frac{2}{5}t^{-5} + C\right) = \frac{2}{5t} + Ct^4$$

$\rightarrow$

$$\Rightarrow v(t) = \frac{2}{5t} + ct^4 = \frac{2+5ct^5}{5t}$$

$$v = y^{-2} \Rightarrow v = \frac{1}{y^2} \Rightarrow y^2 = \frac{1}{v} \Rightarrow y = \pm\left(\frac{1}{v}\right)^{1/2}$$

$$\Rightarrow \boxed{y(t) = \pm\sqrt{\frac{5t}{2+5ct^5}}}$$

10. Riccati Equations

$$\frac{dy}{dt} = q_1(t) + q_2(t)y + q_3(t)y^2$$

(a) suppose that $y_1(t)$ is a solution of the DE above.

$$\text{Let } y(t) = y_1(t) + \frac{1}{v(t)} = y_1(t) + (v(t))^{-1}$$

$$\Rightarrow \frac{dy}{dt} = \frac{dy_1}{dt} - (v(t))^{-2} \cdot \frac{dv}{dt}$$

So for $y(t)$ to be a solution of the Riccati DE, the following must be satisfied:

$$\frac{dy_1}{dt} - v^{-2}\frac{dv}{dt} = q_1 + q_2\left(y_1 + \frac{1}{v}\right) + q_3\left(y_1 + \frac{1}{v}\right)^2$$

$$\frac{dy_1}{dt} - v^{-2}\frac{dv}{dt} = q_1 + q_2 y_1 + q_2 \cdot \frac{1}{v} + q_3 y_1^2 + 2q_3 y_1 \cdot \frac{1}{v} + q_3 v^{-2}$$

$$\Rightarrow \underbrace{\left(\frac{dy_1}{dt} - q_1 - q_2 y_1 - q_3 y_1^2\right)}_{\substack{=0 \text{ since } y_1 \text{ is a} \\ \text{solution of the DE.}}} - v^{-2}\frac{dv}{dt} = q_2 v^{-1} + 2q_3 y_1 v^{-1} + q_3 v^{-2}$$

$$\Rightarrow -v^{-2}\frac{dv}{dt} = q_2 v^{-1} + 2q_3 y_1 v^{-1} + q_3 v^{-2}$$

$$\Rightarrow \frac{dv}{dt} = -q_2 v - 2q_3 y_1 v - q_3$$

$$\Rightarrow \boxed{\frac{dv}{dt} = -(q_2 + 2q_3 y_1)v - q_3},$$ which is a first-order linear DE for $v(t)$.

(b) $\frac{dy}{dt} = 1 + t^2 - 2ty + y^2$

$q_1 = 1+t^2$

$q_2 = -2t$

$q_3 = 1$

$y_1 = t$: $\frac{dy_1}{dt} = 1$

$1+t^2-2ty_1+y_1^2 = 1+t^2-2t^2+t^2 = 1$

$\Rightarrow y_1 = t$ is a particular solution.

let $y(t) = y_1(t) + \frac{1}{v(t)} = t + \frac{1}{v(t)}$

Then the DE becomes

$$\frac{dv}{dt} = -(-2t+2t)v - 1 = -1 \Rightarrow v = -t + C$$

$$\Rightarrow \boxed{y(t) = t + \frac{1}{C-t}}$$

$\rightarrow$

11. Abel's Theorem.

(a) Since y_1 and y_2 are solutions of the DE, we have:

$$y_1'' + p(t)y_1' + q(t)y_1 = 0 \qquad (1)$$

$$y_2'' + p(t)y_2' + q(t)y_2 = 0. \qquad (2)$$

Multiplying the first equation by y_2 and the second by y_1, we obtain:

$$y_1''y_2 + p(t)y_1'y_2 + q(t)y_1y_2 = 0 \qquad (3)$$

$$y_1y_2'' + p(t)y_1y_2' + q(t)y_1y_2 = 0. \qquad (4)$$

Subtracting Eqn. (3) from Eqn. (4), we obtain:

$$(y_1y_2'' - y_1''y_2) + p(t)(y_1y_2' - y_1'y_2) = 0,$$

as needed.

(b) $W = y_1y_2' - y_1'y_2$

$$\Rightarrow W' = y_1y_2'' + y_1'y_2' - y_1'y_2' - y_1''y_2$$

$$= y_1y_2'' - y_1''y_2.$$

Thus, the equation that we derived in part (a) can be rewritten as:

$$W' + p(t)W = 0.$$

(c) From (b), we have:

$$\frac{dW}{dt} = -p(t)W$$

$$\Rightarrow \frac{1}{W}\, dW = -p(t)\, dt$$

$$\Rightarrow \ln|W| = -\int p(t)\, dt$$

$$\Rightarrow |W| = \exp\left[-\int p(t)\, dt\right]$$

$$\Rightarrow W = c\cdot\exp\left[-\int p(t)\, dt\right].$$ ▨

12. $t^2 y'' + 3ty' + y = 0, \ t > 0$

$$y'' + \frac{3}{t}y' + \frac{1}{t^2}y = 0. \qquad p(t) = \frac{3}{t}, \ q(t) = \frac{1}{t^2}$$

(a) $y_1 = t^{-1} \quad y_1' = -t^{-2} \quad y_1'' = 2t^{-3}$

$$\Rightarrow y_1'' + \frac{3}{t}y_1' + \frac{1}{t^2}y_1 = 2t^{-3} + \frac{3}{t}\cdot -t^{-2} + \frac{1}{t^2}\cdot t^{-1}$$

$$= 2t^{-3} - 3t^{-3} + t^{-3}$$

$$= 0 \checkmark$$

Thus, $y_1 = t^{-1}$ is a solution.

(b) suppose y_2 is a second solution.

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2$$

$$= t^{-1} y_2' + t^{-2} y_2$$

$$W(y_1, y_2) = c \cdot \exp\left[-\int p(t)\,dt\right] = c \cdot \exp\left[-\int \frac{3}{t}\,dt\right]$$

$$= c\cdot \exp[-3\ln|t| = c\cdot \exp[\ln \cdot t^{-3}] = ct^{-3}$$

$$\Rightarrow t^{-1}y_2' + t^{-2}y_2 = ct^{-3}$$

$$\Rightarrow \frac{1}{t}\frac{dy_2}{dt} + \frac{1}{t^2}y_2 = c\cdot\frac{1}{t^3}$$

$$\frac{dy_2}{dt} + \frac{1}{t}y_2 = c\cdot\frac{1}{t^2}$$

(c)

This is a linear DE for y_2, so we can solve it using the method of integrating factors.

$$g(t) = \frac{1}{t} \quad b(t) = \frac{c}{t^2}$$

$$\int g(t)\,dt = \ln t \Rightarrow \mu(t) = e^{\ln t} = t$$

$$\Rightarrow y_2(t) = \frac{1}{t}\left(\int t\cdot\frac{c}{t^2}\,dt\right)$$

$$= \frac{1}{t}\left(c\ln|t| + d\right) = \frac{c\cdot\ln|t|}{t} + d\cdot\frac{1}{t}$$

8.5

$$\Rightarrow y_2(t) = K_1 \cdot \frac{1}{t} + K_2 \cdot \frac{\ln t}{t}$$

Thus, $y_2(t) = \ln t \cdot t^{-1}$ is a second solution of the DE.

Conclude: The general solution of the DE is:

$$\boxed{y(t) = K_1 \cdot t^{-1} + K_2 \cdot t^{-1} \cdot \ln t}$$