Math 333
Exam 1 Answers

1. Find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{x - e^{-x}}{y + e^{y}}.$$

Solution.

$$\frac{1}{2}y^2 + e^y = \frac{1}{2}x^2 + e^{-x} + C$$

2. Find the general solution of the differential equation

$$\frac{dy}{dx} = -\frac{4x + 3y}{2x + y}.$$

Solution.

$$|y + x||y + 4x|^2 = C$$

3. Find the general solution of the differential equation

$$\frac{dy}{dt} = -3y + e^{-2t} + 4.$$

Solution.

$$y(t) = e^{-2t} + Ce^{-3t} + \frac{4}{3}$$

4. Find the general solution of the differential equation

$$y'' + \frac{y'}{2} + \frac{y}{16} = 0.$$

Solution.

$$y(t) = k_1 e^{-t/4} + k_2 t e^{-t/4}$$

5. Find the solution of the initial-value problem

$$y'' + 6y = 0, \quad y(0) = 1, \quad y'(0) = -1.$$

Solution.

$$y(t) = \cos(\sqrt{6}t) - \frac{\sqrt{6}}{6}\sin(\sqrt{6}t)$$

6. Find a differential equation of the form

$$ay'' + by' + cy = 0$$

for which the function

$$e^{-t} + 7e^{-2t}$$

is a solution, or explain why no such differential equation exists.

Solution.

$$y'' + 3y' + 2y = 0$$

7. Find a differential equation of the form

$$ay'' + by' + cy = 0$$

for which the function

$$y(t) = 5 + e^{-3t}$$

is a solution, or explain why no such differential equation exists.

Solution.

$$y'' + 3y' = 0$$

8. Find a differential equation of the form

$$ay'' + by' + cy = 0$$

for which the function

$$y(t) = e^{t^2}$$

is a solution, or explain why no such differential equation exists.

Solution. No such differential equation exists.

9. **Bernoulli Equations.** Sometimes it is possible to solve a nonlinear differential equation by making a change of the dependent variable that converts it into a linear equation. The most important such equation has the form

$$\frac{dy}{dt} + p(t)y = q(t)y^n$$

and is called a Bernoulli equation.

(a) Solve Bernoulli's equation when $n = 0$.
Solution. Using the method if integrating factors,

$$y(t) = \frac{1}{\exp(\int p(t)\,dt)} \left(\int \exp\left(\int p(t)\,dt \right) \cdot q(t)\,dt \right).$$

(b) Solve Bernoulli's equation when $n = 1$.
Solution.

$$\ln|y| = \int (q(t) - p(t))\,dt.$$

(c) Show that if $n \neq 0, 1$, then the substitution $v = y^{1-n}$ reduces Bernoulli's equation to a linear equation. This method of solution was found by Leibniz in 1696.

Solution.

$$\frac{dv}{dt} + (1-n)p(t)v = (1-n)q(t)$$

(d) Solve the differential equation

$$t^2\frac{dy}{dt} + 2ty - y^3 = 0, \quad t > 0$$

using the method outlined in (a)-(c).

Solution.

$$y(t) = \pm\sqrt{\frac{5t}{2 + 5Ct^5}}$$

10. **Riccati Equations.** The equation

$$\frac{dy}{dt} = q_1(t) + q_2(t)y + q_3(t)y^2$$

is known as a Riccati equation.

(a) Suppose that some particular solution y_1 of this equation is known. A more general solution containing one arbitrary constant can be obtained through the substitution

$$y = y_1(t) + \frac{1}{v(t)},$$

where $v(t)$ is some function that must be determined. Show that $v(t)$ satisfies the first order *linear* equation

$$\frac{dv}{dt} = -(q_2 + 2q_3y_1)v - q_3.$$

Note that $v(t)$ will contain a single arbitrary constant.

(b) Consider the differential equation

$$\frac{dy}{dt} = 1 + t^2 - 2ty + y^2.$$

Show that $y_1 = t$ is a particular solution. Find the general solution using the method outlined in (a).

Solution.

$$y(t) = t + \frac{1}{C - t}$$

11. **Abel's Theorem.** Suppose that y_1 and y_2 are solutions of the differential equation

$$y'' + p(t)y' + q(t)y = 0.$$

Consider the Wronskian

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2.$$

In this series of exercises, you will prove Abel's theorem:

$$W(y_1, y_2) = c \cdot \exp\left[-\int p(t)\,dt\right],$$

where c is a constant that does not depend on t.

(a) Show that
$$(y_1 y_2'' - y_1'' y_2) + p(t)(y_1 y_2' - y_1' y_2) = 0.$$

(b) Show that
$$W' + p(t)W = 0.$$

Solution.
$$W' = y_1 y_2'' - y_1'' y_2,$$
so the equation in part (a) can be rewritten as
$$W' + p(t)W = 0.$$

(c) Conclude that
$$W = c \cdot \exp\left[-\int p(t)\,dt\right].$$

12. Consider the differential equation

$$t^2 y'' + 3ty' + y = 0, \quad t > 0.$$

(a) Show that $y_1(t) = t^{-1}$ is a solution of the differential equation.

(b) Suppose that y_2 is a second solution of the differential equation. Use Abel's theorem to write a first-order linear differential equation that must be satisfied by y_2.
Solution.
$$\frac{dy_2}{dt} + \frac{1}{t}y_2 = C\frac{1}{t^2}$$

(c) Solve the differential equation in (b) to find a second solution y_2 of the original differential equation.
Solution.
$$y_2(t) = t^{-1}\ln t$$