Math 333
Exam 1

- You have approximately one week to work on this exam. The exam is due at 5:00 pm on Thursday, February 28. No late exams will be accepted.
- During the exam, you are permitted to use your textbook, your class notes, and any lecture notes or other reference material that I have posted on the course website. You may NOT use any other sources during the exam, and you must sign an honor pledge below indicating that you have adhered to these guidelines.
- You are permitted to use Maple (for simulation only) during this exam, and you may, if you wish, use the Maple files that I have posted on the Course Schedule page as a reference.
- You must show all work to receive credit. Answers for which no work is shown will receive no credit (unless specifically stated otherwise).
- Good luck!

Name:

"On my honor, I have neither given nor received any unauthorized aid on this examination."

Signature:

Question	Score	Maximum
1		5
2		5
3		5
4		5
5		5
6		5
7		5
8		5
9		15
10		15
11		15
12		15
Total		100

1. Find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{x - e^{-x}}{y + e^{y}}.$$

2. Find the general solution of the differential equation

$$\frac{dy}{dx} = -\frac{4x + 3y}{2x + y}.$$

3. Find the general solution of the differential equation

$$\frac{dy}{dt} = -3y + e^{-2t} + 4.$$

4. Find the general solution of the differential equation

$$y'' + \frac{y'}{2} + \frac{y}{16} = 0.$$

5. Find the solution of the initial-value problem

$$y'' + 6y = 0, \quad y(0) = 1, \quad y'(0) = -1.$$

6. Find a differential equation of the form

$$ay'' + by' + cy = 0,$$

where a, b, and c are real constants with $a \neq 0$, for which the function

$$e^{-t} + 7e^{-2t}$$

is a solution, or explain why no such differential equation exists.

7. Find a differential equation of the form

$$ay'' + by' + cy = 0,$$

where a, b, and c are real constants with $a \neq 0$, for which the function

$$y(t) = 5 + e^{-3t}$$

is a solution, or explain why no such differential equation exists.

8. Find a differential equation of the form

$$ay'' + by' + cy =, 0$$

where a, b, and c are real constants with $a \neq 0$, for which the function

$$y(t) = e^{t^2}$$

is a solution, or explain why no such differential equation exists.

9. **Bernoulli Equations.** Sometimes it is possible to solve a nonlinear differential equation by making a change of the dependent variable that converts it into a linear equation. The most important such equation has the form

$$\frac{dy}{dt} + p(t)y = q(t)y^n$$

and is called a Bernoulli equation.

(a) Solve Bernoulli's equation when $n = 0$.

(b) Solve Bernoulli's equation when $n = 1$.

(c) Show that if $n \neq 0, 1$, then the substitution $v = y^{1-n}$ reduces Bernoulli's equation to a linear equation. This method of solution was found by Leibniz in 1696.

(d) Solve the differential equation

$$t^2\frac{dy}{dt} + 2ty - y^3 = 0, \quad t > 0$$

using the method outlined in (c).

10. **Riccati Equations.** The equation

$$\frac{dy}{dt} = q_1(t) + q_2(t)y + q_3(t)y^2$$

is known as a Riccati equation.

(a) Suppose that some particular solution y_1 of this equation is known. A more general solution containing one arbitrary constant can be obtained through the substitution

$$y = y_1(t) + \frac{1}{v(t)},$$

where $v(t)$ is some function that must be determined. Show that $v(t)$ satisfies the first order *linear* equation

$$\frac{dv}{dt} = -(q_2 + 2q_3y_1)v - q_3.$$

Note that $v(t)$ will contain a single arbitrary constant.

(b) Consider the differential equation

$$\frac{dy}{dt} = 1 + t^2 - 2ty + y^2.$$

Show that $y_1 = t$ is a particular solution. Find the general solution using the method outlined in (a).

11. **Abel's Theorem.** Suppose that y_1 and y_2 are solutions of the differential equation
$$y'' + p(t)y' + q(t)y = 0.$$
Consider the Wronskian
$$W(y_1, y_2) = y_1y_2' - y_1'y_2.$$
In this series of exercises, you will prove Abel's theorem:
$$W(y_1, y_2) = c \cdot \exp\left[-\int p(t)\,dt\right],$$
where c is a constant that does not depend on t.

    (a) Show that
    $$(y_1y_2'' - y_1''y_2) + p(t)(y_1y_2' - y_1'y_2) = 0.$$
    (b) Show that
    $$W' + p(t)W = 0.$$
    (c) Conclude that
    $$W = c \cdot \exp\left[-\int p(t)\,dt\right].$$

12. Consider the differential equation
$$t^2y'' + 3ty' + y = 0, \quad t > 0.$$

    (a) Show that $y_1(t) = t^{-1}$ is a solution of the differential equation.
    (b) Suppose that y_2 is a second solution of the differential equation. Use Abel's theorem to write a first-order linear differential equation that must be satisfied by y_2.
    (c) Solve the differential equation in (b) to find a second solution y_2 of the original differential equation.