Exam 2 Practice Solutions

1. $\dfrac{dy}{dx} = \dfrac{x}{y}$

(a) $\int y\,dy = \int x\,dx$

$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C_1$

$\boxed{y^2 = x^2 + C}$

(b) $y(2)=1 \Rightarrow 1^2 = 2^2 + C \Rightarrow C = -3$

$\boxed{y^2 = x^2 - 3}$

2. (a) $\displaystyle\lim_{t\to\infty} \int_3^t \frac{\ln x}{x}\,dx \qquad u = \ln x \quad du = \frac{1}{x}\,dx$

$$= \lim_{t\to\infty} \int_3^t u\,du = \lim_{t\to\infty} \frac{1}{2}u^2 \Big|_3^t = \lim_{t\to\infty} \frac{1}{2}(\ln x)^2 \Big|_3^t$$

$$= \lim_{t\to\infty} \underbrace{\frac{1}{2}(\ln t)^2}_{\to\,\infty \text{ as } t\to\infty} - \frac{1}{2}(\ln 3)^2$$

$\boxed{\text{The integral diverges.}}$

(b) $\dfrac{x}{x^5+1} \le \dfrac{x}{x^5} = \dfrac{1}{x^4}$

$\int_1^\infty \frac{1}{x^4}\,dx$ converges b/c $p = 4 > 1$.

So $\int_1^\infty \frac{x}{x^5+1}$ $\boxed{\text{converges}}$ by the Comparison Test.

(c) $\displaystyle\int_0^4 \frac{1}{x-3}\,dx = \lim_{t\to 3^-} \int_0^t \frac{1}{x-3}\,dx + \lim_{s\to 3^+} \int_s^4 \frac{1}{x-3}\,dx$

$\displaystyle= \lim_{t\to 3^-} \ln|x-3|\,\Big|_0^t + \lim_{s\to 3^+} \ln|x-3|\,\Big|_s^4$

The integral $\boxed{\text{diverges}}$ since $\displaystyle\lim_{t\to 3^-} \ln|x-3| = -\infty$.

3. (a) $\displaystyle\lim_{n\to\infty} \frac{\ln n}{n} \overset{L}{=} \lim_{n\to\infty} \frac{1/n}{1} = 0.$

The sequence $\{a_n\}$ converges to 0.

(b) $\displaystyle\lim_{n\to\infty} \left(\tfrac{1}{2}\right)^n = 0$, so the sequence $\{a_n\}$ converges to 0.

(c) $\lim_{n\to\infty} n = \infty$, so the sequence diverges.

(d) $\lim_{n\to\infty} \left(1+\frac{2}{n}\right)^n = L$. $\ln L = \lim_{n\to\infty} n \ln\left(1+\frac{2}{n}\right)$

$$\ln L = \lim_{n\to\infty} \frac{\ln\left(1+\frac{2}{n}\right)}{\frac{1}{n}} \overset{L}{=} \lim_{n\to\infty} \frac{\frac{1}{1+\frac{2}{n}} \cdot \frac{-2}{n^2}}{-\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{2}{1+\frac{2}{n}} = 2. \Rightarrow L = e^2.$$

The sequence converges to e^2.

4. (a) $\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \sum_{n=1}^{\infty} \left[\frac{1/2}{n} - \frac{1/2}{n+2}\right] = \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right)$

$$S_N = \frac{1}{2} \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+2}\right)$$

$S_1 = \frac{1}{2}\left(1-\frac{1}{3}\right)$ $\quad S_2 = \frac{1}{2}\left[\left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)\right]$

$$S_3 = \frac{1}{2}\left[\left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)\right]$$

In general,

$$S_N = \frac{1}{2}\left[1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2}\right]$$

$$\lim_{N\to\infty} S_N = \frac{1}{2}\left[1+\frac{1}{2}\right] = \frac{3}{4}.$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \boxed{\frac{3}{4}.}$$

(b) $$\sum_{k=0}^{\infty} \frac{2^k+3}{5^k} = \sum_{k=0}^{\infty} \left(\frac{2}{5}\right)^k + \sum_{k=0}^{\infty} 3\cdot\left(\frac{1}{5}\right)^k$$

$$= \frac{1}{1-\frac{2}{5}} + \frac{3}{1-\frac{1}{5}} = \boxed{\frac{65}{12}}$$

5. (a) $\lim_{n\to\infty} e^{-1/n} = e^0 = 1 \neq 0$, so the series $\boxed{\text{diverges}}$

by the Test for Divergence.

(b) $\frac{1}{(n+3)^{3/2}} \leq \frac{1}{n^{3/2}}$. $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges b/c $p = \frac{3}{2} > 1$.

Thus $\sum_{n=1}^{\infty} \frac{1}{(n+3)^{3/2}}$ $\boxed{\text{converges}}$ by the Comparison Test.

(c) $\boxed{\text{converges}}$. Use the Integral Test.

(d). $\lim_{k\to\infty} \left| \dfrac{\frac{(k+1)^2}{3^{k+1}}}{\frac{k^2}{3^k}} \right| = \frac{1}{3} < 1$, so the series $\boxed{\text{converges}}$

by the Ratio Test.

(e) $\lim_{n\to\infty} \left| \dfrac{\frac{(n+1)^5}{(n+1)!}}{\frac{n^5}{n!}} \right| = 0 < 1$, so the series $\boxed{\text{converges}}$

by the Ratio Test.

6. converges absolutely.

$\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{k^4} \right| = \sum_{k=1}^{\infty} \frac{1}{k^4}$ converges b/c $p = 4 > 1$.

7. converges conditionally

$\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{k^{2/3}} \right| = \sum_{k=1}^{\infty} \frac{1}{k^{2/3}}$ diverges b/c $p = \frac{2}{3} < 1$.

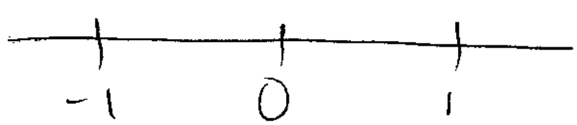
$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{2/3}} = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^{2/3}}$$ converges by

the Alternating Series Test:

(1) $\frac{1}{(k+1)^{2/3}} \le \frac{1}{k^{2/3}}$ and (2) $\lim_{k\to\infty} \frac{1}{k^{2/3}} = 0$.

8. $$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} x^{n+1}}{\sqrt{n+1}}}{\frac{(-1)^n x^n}{\sqrt{n}}} \right| = |x|. \Rightarrow$$

- Ⓒ for $|x| < 1$
- Ⓓ for $|x| > 1$
- ? for $|x| = 1$.



$R = 1$

$\underline{x=1}$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ <u>converges</u> by the AST.

$\underline{x=-1}$: $\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges b/c $p = 1/2 < 1$.

$$\Rightarrow \boxed{I = (-1, 1],\ R = 1}$$

9. $f(x) = \frac{x}{1+x^4} = x \cdot \frac{1}{1+x^4} = x \cdot \frac{1}{1-(-x)^4}$

$= x \cdot \sum_{n=0}^{\infty} (-x^4)^n = x \cdot \sum_{n=0}^{\infty} (-1)^n x^{4n} = \sum_{n=0}^{\infty} (-1)^n x^{4n+1}$

Interval of convergence: $|x^4| < 1 \Rightarrow |x|^4 < 1 \Rightarrow |x| < 1$

$-1 < x < 1$

10. $1 + -1 + 1 + -1 + 1 + -1 + \cdots$ is a geometric series with $r = -1$, which diverges, so any calulations using line 2 are meaningless

11. (a) $\lim_{n \to \infty} b_n = 3 + \frac{2}{5} = \frac{17}{5}$, so the sequence $\{b_n\}$ $\boxed{\text{converges}}$ to $\frac{17}{5}$.

(b) Since $\lim_{n \to \infty} b_n \neq 0$, $\sum_{n=1}^{\infty} b_n$ $\boxed{\text{diverges}}$.