Solutions

1. (5 points) Find the solution of the initial-value problem

$$y' = \frac{e^{-y^2} 2x}{2y}, \quad y(0) = 1.$$

$$\frac{dy}{dt} = \frac{e^{-y^2} \cdot 2t}{2y} \Rightarrow \int 2y e^{y^2} dy = \int 2x \, dx$$

$$e^{y^2} = x^2 + C \quad y(0) = 1 \Rightarrow e^{1^2} = 0^2 + C \Rightarrow C = e$$

$$\boxed{e^{y^2} = x^2 + e}$$

2. Determine whether each of the following improper integrals converges or diverges. (5 points each)

(a) $\int_1^\infty \frac{1+e^{-x}}{x}\,dx$

$\frac{1+e^{-x}}{x} \geq \frac{1}{x}$ $\int_1^\infty \frac{1}{x}\,dx$ diverges b/c $p=1$.

Thus, $\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ $\boxed{\text{diverges}}$ by the Comparison Test.

(b) $\int_1^\infty \frac{1}{(2x+1)^3}\,dx$

$\frac{1}{(2x+1)^3} \leq \frac{1}{x^3}$ $\int_1^\infty \frac{1}{x^3}\,dx$ converges b/c $p=3>1$.

Thus, $\int_1^\infty \frac{1}{(2x+1)^3}\,dx$ $\boxed{\text{converges}}$ by the Comparison Test

Note: This is 2(c).

~~3~~ For each of the following, evaluate the integral or show that it diverges. (5 points each)

(c) ~~(a)~~ $\int_0^3 \frac{1}{x-1}\,dx = \lim_{t\to 1^-} \int_0^t \frac{1}{x-1}\,dx + \lim_{s\to 1^+} \int_s^3 \frac{1}{x-1}\,dx$

$$= \lim_{t\to 1^-} \ln|x-1| \Big|_0^t + \lim_{s\to 1^+} \ln|x-1| \Big|_s^3$$

$$= \underbrace{\lim_{t\to 1^-} \ln|t-1|}_{\to -\infty} + \ln 1 + \ln 2 - \lim_{s\to 1^+} \ln|s-1|$$

The integral $\boxed{\text{diverges}}$

(b) $\int_1^\infty \frac{1}{x(\ln x)^4}\,dx$ Note: this question was not on Exam 2.

The integral is improper at $x=1$ and ∞.

$$\int_1^\infty \frac{1}{x(\ln x)^4}\,dx = \lim_{t\to 1^+} \int_t^2 \frac{1}{x(\ln x)^4}\,dx + \lim_{s\to\infty} \int_2^s \frac{1}{x(\ln x)^4}\,dx \qquad u = \ln x \quad du = \frac{1}{x}\,dx$$

$$= \lim_{t\to 1^+} \int_t^2 u^{-4}\,du + \lim_{s\to\infty} \int_2^s u^{-4}\,du = \lim_{t\to 1^+} -\frac{1}{3}u^{-3} \Big|_t^2 + \lim_{s\to\infty} -\frac{1}{3}u^{-3} \Big|_2^s$$

$$= \lim_{t\to 1^+} -\frac{1}{3}\cdot\frac{1}{(\ln x)^3} \Big|_t^2 + \lim_{s\to\infty} -\frac{1}{3}\cdot\frac{1}{(\ln x)^3} \Big|_2^s$$

The integral $\boxed{\text{diverges}}$

$$= -\frac{1}{3}\cdot\frac{1}{(\ln 2)^3} - \overbrace{\lim_{t\to 1^+} -\frac{1}{3}\frac{1}{(\ln t)^3}}^{\to -\infty} + \lim_{s\to\infty} -\frac{1}{3}\frac{1}{(\ln s)^3} + \frac{1}{3}\frac{1}{(\ln 2)^3}$$

3~~1~~. Determine whether each of the sequences $\{a_n\}$ converges or diverges. If the sequence converges, find its limit. (5 points each)

(a) $a_n = \dfrac{3^{2n}}{5^{n+1}}$

$$\lim_{n\to\infty} \frac{9^n}{5\cdot 5^n} = \frac{1}{5} \lim_{n\to\infty} \left(\frac{9}{5}\right)^n = \infty$$

the sequence diverges.

(b) $a_n = (-1)^{n-1}\dfrac{n}{n^2+1}$

$$\frac{-n}{n^2+1} \le \frac{(-1)^{n-1}n}{n^2+1} \le \frac{n}{n^2+1}$$

$$\frac{-n}{n^2+1} \to 0 \qquad \frac{n}{n^2+1} \to 0$$

$$\lim_{n\to\infty} \frac{(-1)^{n-1}n}{n^2+1} = 0$$

by the Squeeze theorem.

The sequence converges to 0.

(c) $a_n = \dfrac{\ln n}{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} \overset{L}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{2}\cdot n^{-1/2}} = \lim_{n\to\infty} \frac{2\sqrt{n}}{n} = 0.$$

the sequence converges to 0.

4. (10 points) Find the sum of the series $\sum_{k=0}^{\infty} \frac{2^{2k+1}}{5^k}$.

$$= 2 + \frac{2^3}{5} + \frac{2^5}{5^2} + \frac{2^7}{5^3} + \cdots$$

The series is a geometric series with

$a=2$ and $r = 2^2/5$.

Since $|2^2/5| = |4/5| < 1$, the series converges

and is equal to $\frac{a}{1-r}$.

$$\sum_{k=0}^{\infty} \frac{2^{2k+1}}{5^k} = \frac{2}{1-4/5} = \frac{2}{\frac{1}{5}} = \boxed{10}$$

5 ~~4~~. Determine whether each of the following series converges or diverges. (5 points each)

(a) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ Use the Ratio Test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2 \cdot 2^n}{n^2 \cdot 2^{n+1}}\right| = \frac{1}{2} < 1,$$

so the series $\boxed{\text{converges}}$ by the Ratio Test.

(b) $\sum_{n=1}^{\infty} \ln\left(\frac{n}{3n+1}\right)$

$\lim_{n\to\infty} \ln\left(\frac{n}{3n+1}\right) = \ln\left(\frac{1}{3}\right) \neq 0$, so the series

$\boxed{\text{diverges}}$ by the Test for Divergence

(c) $\sum_{k=1}^{\infty} \frac{k-1}{k^3+2}$

$\frac{k-1}{k^3+2} \leq \frac{k}{k^3} = \frac{1}{k^2}$. $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges because $p=2>1$. Thus, $\sum_{k=1}^{\infty} \frac{k-1}{k^3+2}$ $\boxed{\text{converges}}$ by the Comparison Test.

(d) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{\sqrt{k+1}}$ This is an alternating series, so we'll use the AST.

$c_k = \frac{1}{\sqrt{k+1}}$

(i)

$c_{k+1} = \frac{1}{\sqrt{k+1+1}} = \frac{1}{\sqrt{k+2}} \leq \frac{1}{\sqrt{k+1}} = c_k$ ✓

(ii) $\lim_{k \to \infty} \frac{1}{\sqrt{k+1}} = 0$ ✓

Thus, $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{\sqrt{k+1}}$ $\boxed{\text{converges}}$ by the AST.

6~~X~~. (10 points) Find the radius and interval of convergence of the series $\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{n^2 5^n}$.

Start by using the Ratio Test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}x^{n+1}}{(n+1)^2 5^{n+1}}}{\frac{(-1)^n x^n}{n^2 5^n}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}n^2 5^n}{(-1)^n x^n (n+1)^2 5^{n+1}}\right|$$

$$= \left|\frac{x}{5}\right| \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2 = \left|\frac{x}{5}\right|$$

$\Rightarrow$ • C if $\left|\frac{x}{5}\right| < 1 \to |x| < 5 \to -5 < x < 5 \to \boxed{R = 5}$

• D if $\left|\frac{x}{5}\right| > 1 \to |x| > 5$

• ? if $\left|\frac{x}{5}\right| = 1 \to x = \pm 5$

$\underline{x=5}$: $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ converges absolutely ($\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges b/c $p=2$).

$\underline{x=-5}$: $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges b/c $p=2>1$ $\Rightarrow \boxed{I = [-5, 5]}$

~~6~~ 7. (10 points) Find the power series representation of the function

$$f(x) = \frac{x^2}{1+x}.$$

Find the interval of convergence of the power series.

$$f(x) = \frac{x^2}{1+x} = x^2 \cdot \frac{1}{1-(-x)} = x^2 \cdot \sum_{k=0}^{\infty} (-x)^k$$

$$= \sum_{k=0}^{\infty} (-1)^k x^{k+2}$$

interval of convergence: $|-x| < 1 \rightarrow -1 < x < 1$

$I = (-1, 1)$

8 ~~7~~. Give an example of each of the following, or explain why no such example exists. Be sure to justify why the example that you give is appropriate. (5 points each)

(a) A sequence $\{a_n\}$ such that $\lim_{n\to\infty} a_n = 0$ but $\sum_{n=1}^{\infty} a_n$ diverges.

$a_n = \frac{1}{n}$

(b) A sequence $\{a_n\}$ such that $\lim_{n\to\infty} a_n = 1$ but $\sum_{n=1}^{\infty} a_n$ converges.

No such example exists (by the Test for Divergence: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges).

(c) A series $\sum_{k=0}^{\infty} a_k$ that converges to 9π.

Ex 1: $a_0 = 9\pi$, $a_k = 0$ for $k = 1, 2, 3, \ldots$

Ex2: Geometric series with $a = 1$, $r = 1 - \frac{1}{9\pi}$:

$$\sum_{k=0}^{\infty} \left(1 - \frac{1}{9\pi}\right)^k$$

Bonus (5 points). The terms of a series are defined recursively by the equations

$$a_1 = 2, \quad a_{n+1} = \frac{2+\cos(3n)}{n^{2/3}} a_n.$$

Determine whether $\sum_{n=1}^{\infty} a_n$ converges or diverges.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(2+\cos(3n))a_n}{n^{2/3}}}{a_n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{2+\cos(3n)}{n^{2/3}} \right| = 0 < 1,$$ so the series converges by the Ratio Test.

$$0 \le \frac{|2+\cos(3n)|}{n^{2/3}} \le \frac{3}{n^{2/3}}$$

$$0 \xrightarrow{n\to\infty} 0, \qquad \frac{3}{n^{2/3}} \xrightarrow{n\to\infty} 0$$

so $\lim_{n\to\infty} \left| \frac{2+\cos(3n)}{n^{2/3}} \right| = 0$ by the Squeeze Theorem.