Exam 1

Name: Solutions

- You have two hours to complete this exam.
- No notes, books, or other references are allowed.
- You are NOT allowed to use Maple or any other computing resources (including calculators) on this exam.
- You must show all work to receive credit. Answers for which no work is shown will receive no credit (unless specifically stated otherwise).
- Good luck! Eat candy as necessary.

The Error Bound Theorem Formulas

- $|I - L_n| \leq \frac{K_1(b-a)^2}{2n}$ and $|I - R_n| \leq \frac{K_1(b-a)^2}{2n}$
- $|I - M_n| \leq \frac{K_2(b-a)^3}{24n^2}$ and $|I - T_n| \leq \frac{K_2(b-a)^3}{12n^2}$

Question	Score	Maximum
1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total		100

1. Evaluate the following integrals. (5 points each)

(a) $\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$

$u = \sqrt{x}$ $du = \frac{1}{2\sqrt{x}}\,dx$

$2du = \frac{1}{\sqrt{x}}\,dx$

$\int 2\sin u\,du$

$= -2\cos u + C \to \boxed{-2\cos(\sqrt{x}) + C}$

(b) $\int \frac{x+2}{x^2-4x}\,dx$

$\frac{x+2}{x^2-4x} = \frac{A}{x} + \frac{B}{x-4}$ $A = -1/2,\ B = 3/2.$

$\int \frac{x+2}{x^2-4x}\,dx = \boxed{-\frac{1}{2}\ln|x| + \frac{3}{2}\ln|x-4| + C}$

(c) $\int x^2 \ln x \, dx$

$u = \ln x \quad dv = x^2 dx$

$du = \frac{1}{x} dx \quad v = \frac{1}{3}x^3$

$= \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^3 \cdot \frac{1}{x} \, dx$

$= \boxed{\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C}$

(d) $\int_0^1 e^{2x} e^{e^x} \, dx$

$w = e^x \quad dw = e^x dx$

$= \int_{x=0}^{x=1} (e^x)^2 e^{e^x} dx = \int_{x=0}^{x=1} e^x e^x e^{e^x} dx = \int_{x=0}^{x=1} w e^w dw$

$u = w \quad dv = e^w dw$

$du = dw \quad v = e^w$

$= we^w - \int_{x=0}^{x=1} e^w dw$

$= we^w - e^w \Big]_{x=0}^{x=1}$

$= e^x e^{e^x} - e^{e^x} \Big]_{x=0}^{x=1} = \boxed{e^e(e-1) = e^{1+e} - e^e}$

2. In each of the following, give an example of a function f that satisfies the given equality, or explain why no such function exists. (5 points each)

(a) $\int_0^2 f(x)\,dx = \frac{9\pi}{2}$

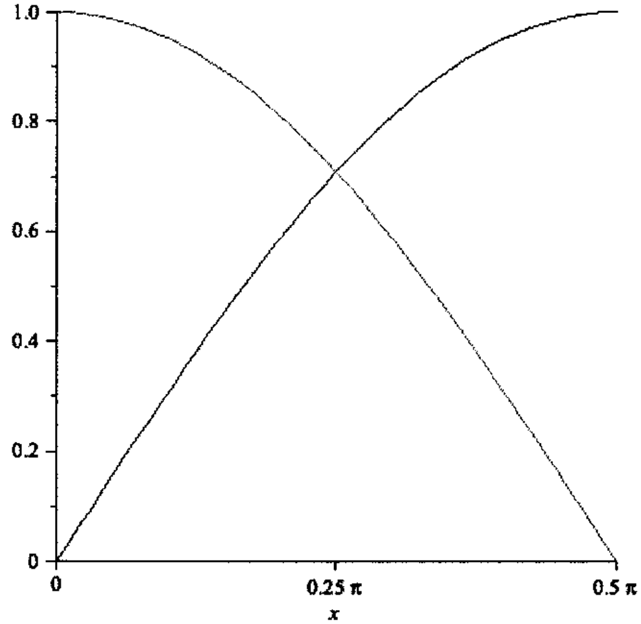
$f(x) = \frac{9\pi}{4}$

$\int \frac{9\pi}{4}\,dx = \frac{9\pi}{4} x \Big|_0^2 = \frac{9\pi}{2}$

(b) $\int f(x)\,dx = e^{x^2} + C$

$f(x) = 2x e^{x^2}$

3. Let R be the region below both $f(x) = \cos x$ and $g(x) = \sin x$ and above the x-axis with $0 \le x \le \frac{\pi}{2}$.



(a) (5 points) Set up, *but do not evaluate*, an integral expression to find the volume obtained by rotating R around the x-axis. You do not need to evaluate the integrals(s).

$$V = \int_0^{\pi/4} \pi (\sin x)^2 \, dx + \int_{\pi/4}^{\pi/2} \pi (\cos x)^2 \, dx$$

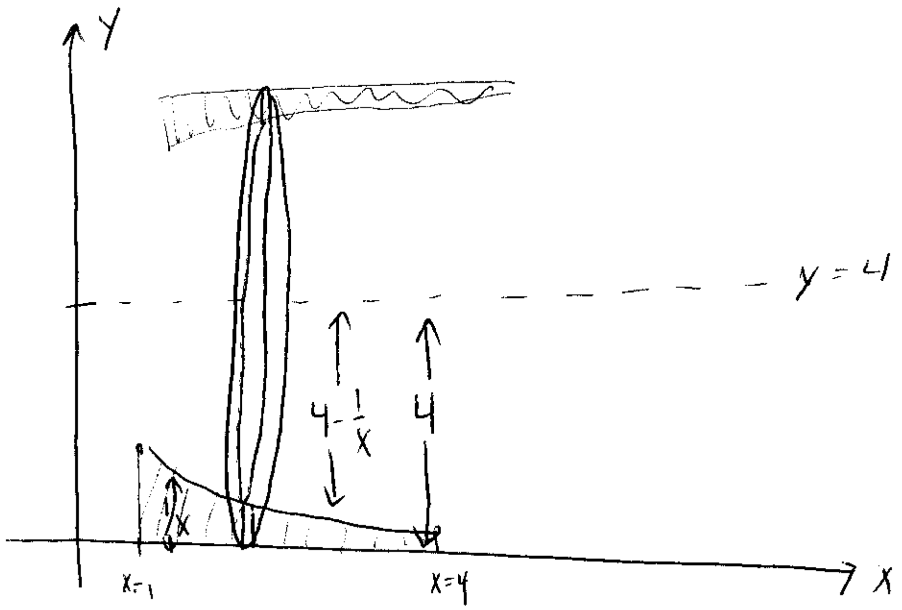
(b) (5 points) Set up, *but do not evaluate*, an integral expression to find the length of the boundary of R. You do not need to evaluate the integral(s).

$$L = \frac{\pi}{2} + \int_0^{\pi/4} \sqrt{1 + (\cos x)^2}\, dx + \int_{\pi/4}^{\pi/2} \sqrt{1 + (-\sin x)^2}\, dx$$

4. (10 points) Set up, *but do not evaluate*, an integral expression to find the volume of the solid generated by revolving the region bounded by the graphs of

$$y = \frac{1}{x}, \quad y = 0, \quad x = 1, \text{ and } x = 4$$

about the line $y = 4$.



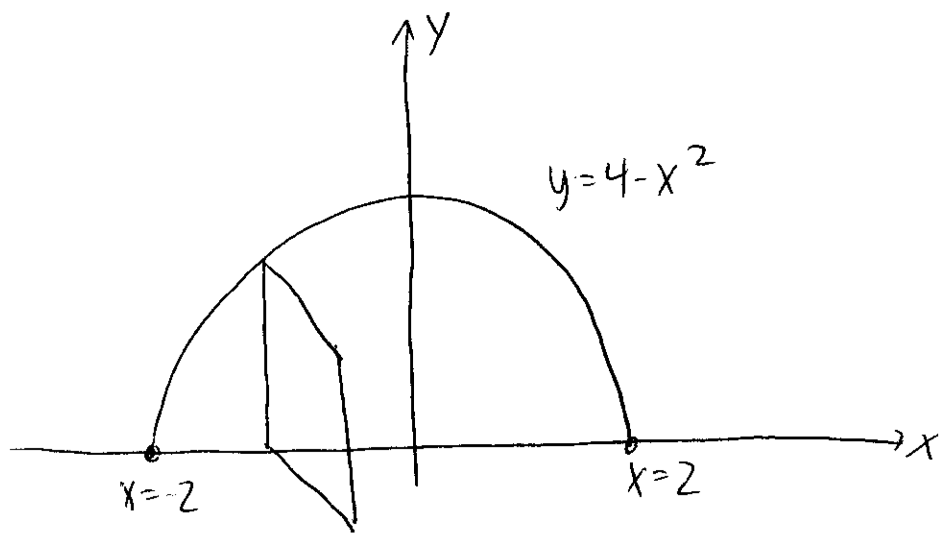
Cross-sections are washers

$R(x) = 4$

$r(x) = 4 - \frac{1}{x}$

$$V = \int_{x=1}^{x=4} \left[\pi(4)^2 - \pi\left(4 - \frac{1}{x}\right)^2 \right] dx$$

5. (10 points) The base of a solid is the region bounded by the graphs of $y = 4 - x^2$ and $y = 0$. Cross-sections of the solid perpendicular to the x-axis are squares. Set up, *but do not evaluate*, an integral expression to find the volume of the solid.



$$A(x) = (\text{side})^2 = (4-x^2)^2$$

$$V = \int_{-2}^{2} (4-x^2)^2 \, dx$$

6. Evaluate each of the following limits. (5 points each)

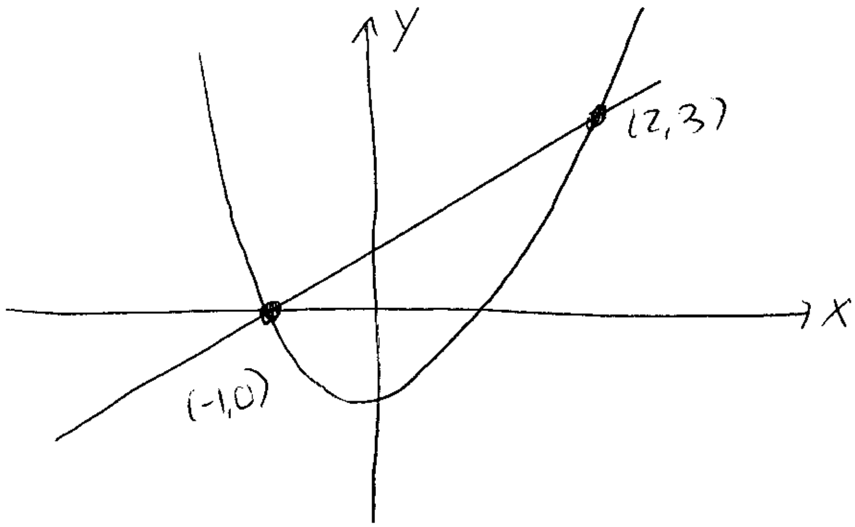
(a) $\lim_{x\to 0} \frac{e^{2x}-1}{x}$. $= \frac{0}{0} \overset{L}{=} \lim_{x\to 0} \frac{2e^{2x}}{1} = \boxed{2}$

(b) $\lim_{x\to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$.

$= \lim_{x\to 1^+} \frac{(x-1) - \ln x}{(\ln x)(x-1)} = \frac{0}{0} \overset{L}{=} \lim_{x\to 1^+} \frac{1 - 1/x}{\ln x + \frac{x-1}{x}}$

$= \frac{0}{0} \overset{L}{=} \lim_{x\to 1^+} \frac{1/x^2}{\frac{1}{x} + \frac{1}{x^2}} = \boxed{\frac{1}{2}}$

7. (10 points) Find the area of the region bounded by the graphs of $y = x^2 - 1$ and $y = x + 1$.



$$A = \int_{-1}^{2} \left[(x+1) - (x^2 - 1) \right] dx$$

$$= \int_{-1}^{2} (-x^2 + x + 2)\, dx = -\frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x \Big|_{-1}^{2}$$

$$= \boxed{\frac{9}{2}}$$

8. (10 points) Set up (but do not evaluate) an inequality to determine a value of n such that the trapezoid rule approximates

$$I = \int_1^2 \cos(1 + x^2)\, dx$$

with error guaranteed to be less than 0.0001. Your final answer should be in the form $n \geq$ stuff, where the stuff contains only numbers. No simplification is necessary.

$f(x) = \cos(1+x^2)$ $\quad |I - T_n| \leq \dfrac{K_2 (2-1)^3}{12n^2} \leq 0.0001$

$|f''(x)| \leq K_2$ on $[1,2]$

$f'(x) = -2x \sin(1+x^2)$

$f''(x) = -2\sin(1+x^2) - 4x^2\cos(1+x^2)$

$|f''(x)| \leq |2\sin(1+x^2)| + |4x^2\cos(1+x^2)|$

$\leq 2 + 16 = 18$. Use $K_2 = 18$.

$$\Rightarrow n \geq \sqrt{\frac{18 \cdot 1^3}{12 \cdot 0.0001}}$$

9. (10 points) Suppose that f is positive, increasing, and concave down on the interval $[1, 7]$. Let

$$I = \int_1^7 f(x)\, dx.$$

Rank the values I, L_{100}, M_{100}, R_{100}, T_{100} in increasing order.

$L_{100} \leq T_{100} \leq I \leq M_{100} \leq R_{100}$.