Putnam Competition

1. **(P00)** The octagon $P_1P_2P_3P_4P_5P_6P_7P_8$ is inscribed in a circle, with the vertices around the circumference in the given order. Given that the polygon $P_1P_3P_5P_7$ is a square of area 5, and the polygon $P_2P_4P_6P_8$ is a rectangle of area 4, find the maximim possible area of the octagon.

2. **(P01)** Can an arc of a parabola inside a circle of radius 1 have a length greater than 4?

3. **(P04)** Basketball star Shanille O'Keal's team statistician keeps track of the number, $S(n)$, of successful free throws she made in her first N attempts of the season. Early in season, $S(n)$ was less than 80% of N, but by the end of the season, $S(N)$ was more than 80% of N. Was there necessarily a moment in between when $S(N)$ was exactly 80% of N?

4. **(P08)** Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled.

   Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero.

   Which player has a winning strategy?

5. **(P08)** Let $F_0(x) = \ln(x)$. For $n \geq 0$ and $x > 0$, let $F_{n+1}(x) = \int_0^x F_n(t)dt$. Evaluate

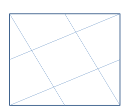
$$\lim_{n\to\infty} \frac{n! F_n(1)}{\ln n}.$$

6. **(P07)** Let f be a nonconstant polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n)+1)$ if and only if $n = 1$.

7. In how many zeroes does 10000! end?

8. What is the sum of all digits used in writing down the numbers from one to a billion?

9. Lines are drawn from the vertices of a square to the midpoints of the sides as shown below. What is the ratio of the original square to the area of the center square S? Can you solve this problem without making any arithmetic or algebraic calculations?



10. Explain the rule which generates the following sequence:
    $2, 3, 10, 12, 13, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 200, 201, 202, ..$
    Hint: Don't think mathematically.

11. In a round-robin tournament with n players $P_1, P_2, \ldots, P_n$, where $n > 1$, each player plays one game with each of the other players and rules are such that no ties can occur. Let W_r and L_r be the number of games won and lost, respectively, by player P_r. Show that

$$\sum_{r=1}^{n} W_r^2 = \sum_{r=1}^{n} L_r^2.$$

12. Prove that any subset of 55 numbers chosen from the set $\{1, 2, 3, 4, \ldots, 100\}$ must contain 2 numbers differing by 9.

13. (P03) A Dyck n-path is a lattice path of n upsteps $(1,1)$ and n downsteps $(1,-1)$ that starts at the origin O and never dips below the x-axis. A return is a maximal sequence of contiguous downsteps that terminates on the x-axis. For example, the Dyck 5-path illustrated has 2 returns, of length 3 and 1, respectively. Show that there is a one-to-one correspondence between the Dyck n-paths with no return of even length and the Dyck $(n-1)$-paths.