Computing the Projective Indecomposable Modules of Large Finite Groups

by

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Abstract

Let G be a finite group and $\mathbb{F}$ be a finite field. A projective indecomposable $\mathbb{F}G$-module is an indecomposable direct summand of the group algebra $\mathbb{F}G$. Computing the projective indecomposable modules of large finite groups has been always a challenging problem due to the large sizes of the representations of these groups. This dissertation describes a new algorithm for constructing the projective indecomposable modules of large finite groups. This algorithm uses the condensation techniques as described in [12]. The power of the algorithm will be illustrated by the examples of the socle series of all projective indecomposable modules of the sporadic simple Mathieu group M_{24} and the simple alternating group A_{12} in characteristic 2.

INTRODUCTION

The main aim of representation theory is the study of possible ways to realize a group as a matrix group. More specifically, a representation ρ of a finite group G over a field $\mathbb{F}$ is a homomorphism $\rho\colon G \to GL(V)$ of G into the group $GL(V)$ of invertible $\mathbb{F}$-endomorphisms of a finite dimensional vector space V over $\mathbb{F}$. Any representation ρ of a finite group G over a field $\mathbb{F}$ extends to a representation of the group algebra $\mathbb{F}G$, $\rho\colon \mathbb{F}G \to \mathrm{End}(V)$ and the vector space V is called an $\mathbb{F}G$-module. Originally the starting point of representation theory was the theory of representations of finite groups over the complex numbers studied by G. Frobenius. Frobenius introduced the traces of the matrices of group elements, which he called characters. He showed that the character of a representation gives a lot of information about the matrix representation itself. Later E. Noether used the notion of the group algebra to unify the structure theory of algebras and the representation theory of finite groups. This approach had an essential impact on the development of the representation theory of finite groups.

In the 1930's, R. Brauer extended the representation theory of finite groups from fields of characteristic zero to fields of finite characteristic. For a given representation over a field of characteristic p, p a prime, Brauer defined a map from the conjugacy classes of elements whose orders are coprime to p, into the complex numbers, which he called the p-modular character of the representation. Another common name for such a character in the literature is Brauer character. The Brauer characters became one of the central tools for studying the representations of a finite group in finite characteristic.

In the late 1950's K. Morita observed that from a categorical point of view the representations of a finite dimensional algebra can be linked to the representations of a usually much smaller algebra. The relationship defined by Morita between the representations of two algebras is now called a Morita equivalence and it establishes a functorial equivalence between the module categories of the two algebras. One of the topics of this dissertation is the use of Morita's theory as a computational tool in the representation theory of finite groups.

Around 1970's computational aspects of representation theory of finite groups became more essential due to the classification of finite simple groups project. This project forced the development of new algorithms in representation theory. After the classification had been established a further task was studying properties of the finite simple groups, such as their irreducible modular representations. The use of Morita equivalence became neseccary since the irreducible representations of finite simple groups tend to be too large to be studied directly. The condensation method, first described by J. Thackray in his Ph.D. thesis, suggests to study the representations of a subalgebra of the group algebra in question, which can be chosen to be Morita equivalent to the given group algebra. Thackray called his technique fixed point condensation. Since fixed point condensation reduces the dimension of the representations one has to study considerably it has turned out to be a very useful tool.

A more sophisticated condensation technique called the peakword condensation was introduced by K.Lux, J. Müller and M. Ringe in [14]. In their method the functorial relationship between module categories is exploited in a new direction, rather than aiming at a Morita equivalence. If we are given a matrix representation of a finite dimensional algebra A one can show that the kernel of a well chosen element in A carries structural information about the representation itself. More precisely, a suitable element of the given algebra A, whose matrix is singular, called a peakword, leads to a primitive idempotent in A. The images of such idempotents can then be used to recover the whole lattice of A-invariant submodules in the underlying module.

Fixed point condensation together with the peakword condensation are previously used for constructing the indecomposable direct summands of the regular $\mathbb{F}G$-

modules, which are called the **projective indecomposable modules** of G. Knowledge of the structure of projective indecomposable modules is not only of interest to modular representation theory but also important for the study of the cohomology of groups. In [15], K. Lux and M. Wiegelmann constructed the socle series of the projective indecomposable modules of the Mathieu group M_{23} over the field with two elements using these condensation techniques. However computing the projective indecomposable modules for large finite groups is a challenging problem due to the large size of the representations of these groups. For example the projective indecomposable modules of the largest Mathieu group M_{24} in characteristic 2 had not been constructed due to the following difficulty: The order of M_{24} is 244823040. Storing a matrix for the regular representation of M_{24} over the field with characteristic two would naively require approximately 56 petabytes (approximately 10^{15} bytes), far beyond the capabilities of contemporary machines. In this dissertation I have developed a new technique for computing the projective indecomposable modules of large finite groups and implemented it in the computer algebra system GAP. Not only is this method more efficient than previous approaches but it may be successfully applied to larger groups. For example, I have used my technique to find the previously unknown projective indecomposable modules of the largest sporadic Mathieu group M_{24} and of the alternating group A_{12} over the field $\mathbb{F}$ with characteristic two. We should note here that M_{24} was the last sporadic simple group with order less than 1000000000 whose projective indecomposable modules over the field with characteristic 2 was not known. My results allow for further analysis of large finite simple groups and their representations. These results also have direct applications in other areas of Algebra; for example, in [3], the authors describe a scheme for computing the Ext-algebra and cohomology ring for a small simple group which is dependent upon knowing the projective indecomposable modules for this group. Similarly computing the cohomology groups of FG-modules via projective resolutions depends on the projective indecomposable modules of G. My method for calculating the projective indecomposable modules presents an opportunity to extend this and other techniques to larger groups.

The algorithm developed in this dissertation for constructing the projective indecomposable modules of a large finite group G has two main steps:

The first step is to construct a subalgebra $e\mathbb{F}Ge$ of the group algebra $\mathbb{F}G$ that is Morita equivalent to $\mathbb{F}G$, where e is an idempotent in $\mathbb{F}G$. Let $S_1, \ldots, S_r$ be the composition factors up to isomorphism of an $\mathbb{F}G$-module V. If e is an idempotent in $\mathbb{F}G$ such that $S_i e \neq 0$, for all $1 \leq i \leq r$, then e is called a faithful idempotent. Faithful idempotents are easy to find in the group algebra $\mathbb{F}G$. For a certain subgroup H of G, where the characteristic p of $\mathbb{F}$, does not divide the order of H, the element $e_H := \frac{1}{|H|}\sum_{h\in H} h$ is an idempotent in $\mathbb{F}G$. If $S_i e_H \neq 0$, for all $1 \leq i \leq r$, the group algebra $\mathbb{F}G$ is Morita equivalent to the algebra $e_H\mathbb{F}Ge_H$, called the condensation subalgebra of $\mathbb{F}G$ and H is called a faithful condensation subgroup of G. The $e_H\mathbb{F}Ge_H$-module Ve_H consists of the fixed points of the action of H on the regular $\mathbb{F}G$-module V, hence the dimension of Ve_H is usually much smaller than the dimension of V. However, any information that we obtain about Ve_H gives rise to information about V via Morita equivalence, for example the projective indecomposable summands of the regular $\mathbb{F}G$-module V is in one to one correspondence with the projective indecomposable direct summands of the regular $e_H\mathbb{F}Ge_H$-module Ve_H. It is important here to note that the matrices corresponding to the elements of the condensed algebra $e_H\mathbb{F}Ge_H$ have at most $|H|$ many nonzero entries per row. Such matrices are called row sparse. In this dissertation, I defined a row sparse matrix format in the computer algebra system GAP to reduce the required memory in my computations. Recalling the M_{24} example over the field $\mathbb{F}$ with characteristic two, let H be a faithful condensation subgroup of M_{24} of order 27. Then the algebra $e_H F M_{24} e_H$ has dimension 336224, much smaller than 244823040. However the regular $e_H F M_{24} e_H$-module is still out of reach of what would be reasonable to work with on a computer today. On the other hand storing one row sparse matrix corresponding to the action of an element $e_H g e_H \in e_H\mathbb{F}Ge_H$

in Ve_H with at most 27 nonzero entries per row requires only 45 megabytes.

The second step of the main algorithm of this dissertation is applying a modified version of the peakword condensation to several projective $e_H\mathbb{F}Ge_H$-module Ve_H that are constructed in the first step. The indecomposable summands of the module Ve_H can be recovered by the stable kernel of the peakwords in $\mathbb{F}G$ in the module Ve_H. There are existing implementations that calculate the peakwords with respect to Ve_H and find a basis for the stable kernel of these peakwords in Ve_H. However these implementations are limited in the size of representations which they can handle. In my dissertation, I create and implemented algorithms that compute a single vector in the stable kernel of a peakword in Ve_H, instead of computing a basis for the stable kernel of a peakword in Ve_H.

Finding the projective indecomposable modules for large simple groups using the algorithm developed in this dissertation is an ongoing project. For example, the projective indecomposable modules for the McLaughlin group McL in characteristic 3 and for the Held group He in characteristics 2, 3 and 5 are still not known and can be found using this algorithm.

As mentioned earlier the computational techniques described in this dissertation have applications in homological algebra, such as finding Ext-algebras, cohomology ring for large group algebras and computing the cohomology groups of $\mathbb{F}G$-modules. I would like to find the unknown Ext-algebras and cohomology ring for large group algebras over fields of various characteristics. The technique described in [3] for finding the Ext-algebras of group algebras $\mathbb{F}G$ requires finding the basic algebra (i.e., the smallest algebra that is Morita equivalent to $\mathbb{F}G$) for G and computing the projective resolutions for simple $\mathbb{F}G$-modules. Both are dependent upon knowing the projective indecomposable modules for G. The algorithm described in this dissertation for computing the projective indecomposable modules of large finite groups, enables one to compute basic algebras for larger groups. The basic algebras of M_{24} and A_{12} in characteristic 2 are still not computed, but can be computed with the constructed projective indecomposable modules of M_{24} and A_{12} in characteristic 2, respectively.

This dissertation is organized as follows:

Chapter 1 covers the background material necessary for the terminology contained in this dissertation. Of particular importance are Sections 1.2 and 1.3: they contain the definitions of a projective indecomposable module and certain idempotents of an algebra which are essential in this dissertation. In this chapter, we also discuss the main facts from the ordinary character theory and the theory of Brauer characters.

In Chapter 2 we consider an algorithm for proving the irreducibility of a given module for a finite dimensional algebra A. We will introduce the Norton criterion, which is the essential tool for proving the irreducibility of a given A-representation. We will then discuss its generalization due to D. Holt and S. Rees.

Chapter 3 covers the basics of Morita theory, including both the consequences of Morita equivalence and the necessary conditions for two algebras to be Morita equivalent. We will then analyze the important case of group algebras. Moreover, a constructive method is presented that finds an algebra that is in general Morita equivalent to a given group algebra. The theory of peakword condensation is introduced also in this chapter. Finally, we describe an algorithm for finding the projective indecomposable modules of a finite group G that is in the GAP data library, which uses both fixed point and peakword condensation.

In Chapter 4 we introduce a new technique for computing the projective indecomposable modules of a large finite group. We will discuss a sparse matrix format in GAP and its use in the implementations of our algorithms.

Chapter 5 describes the results achieved by the implementation discussed in Chapter 4. The socle series of the projective indecomposable modules of the Mathieu group M_{24} and the alternating group A_{12} in characteristic 2 are displayed in this chapter.

Chapter 1
BACKGROUND

The purpose of this chapter is to introduce the main concepts and results which are used in this dissertation. Most of the statements in this chapter are well known and can also be found in the standard textbooks on representation theory, see for example [4], [5].

1.1 Algebras and Modules

The main object of study in this dissertation are finite dimensional algebras and their representations. This section will give the basic definitions necessary for this study.

Definition 1.1.1. *Let $\mathbb{F}$ be a field. An* ***algebra*** *A* ***over*** *$\mathbb{F}$ is a ring A with an identity element, 1_A, which is at the same time an $\mathbb{F}$-vector space such that*

$$\alpha(ab) = (\alpha a)b = a(\alpha b),$$

for $\alpha \in \mathbb{F}$, and $a, b \in A$.

An algebra can be defined more generally over a commutative ring instead of a field. However since all the algebras considered in this dissertation are over fields we use Definition 1.1.1. We will also restrict ourselves to the study of finite dimensional algebras over some field $\mathbb{F}$.

Example 1.1.2. *Let $\mathbb{F}$ be a field. For a positive integer n, the set of all n by n matrices over $\mathbb{F}$ forms an algebra over $\mathbb{F}$ with respect to the usual matrix addition, matrix multiplication and scalar multiplication. This matrix algebra is denoted by $M_n(\mathbb{F})$.*

Example 1.1.3. *Let $\mathbb{F}$ be a field and V an $\mathbb{F}$-vector space. The ring of $\mathbb{F}$-linear endomorphisms* $\mathrm{End}_{\mathbb{F}}(V)$ *of V is an algebra over $\mathbb{F}$.*

Definition 1.1.4. *A subring of an algebra A over a field $\mathbb{F}$ with an identity element, 1_A, which is also an $\mathbb{F}$-subspace of A is called a **subalgebra** of A.*

A central example for an algebra over a field $\mathbb{F}$ in this dissertation is the group algebra of a finite group G over $\mathbb{F}$.

Definition 1.1.5. *Let $\mathbb{F}$ be a field and G be a finite group. The **group algebra** $\mathbb{F}G$ is an $\mathbb{F}$-vector space with basis G and with a multiplication which extends $\mathbb{F}$-linearly the multiplication in G. The elements of $\mathbb{F}G$ have the form*

$$\sum_{g\in G} \alpha_g g, \quad \alpha_g \in \mathbb{F}, g \in G$$

and the multiplication in $\mathbb{F}G$ is defined as

$$(\sum_{g\in G} \alpha_g g) \cdot (\sum_{h\in G} \beta_h h) = \sum_{g,h\in G} \alpha_g \beta_h gh,$$

for $\alpha_g, \beta_h, \gamma_t \in \mathbb{F}$.

Definition 1.1.6. *Let A and B be algebras over a field $\mathbb{F}$. An **$\mathbb{F}$-algebra homomorphism** T from A to B is a map*

$$T \colon A \to B$$

that satisfies

1. $T(a+b) = T(a) + T(b)$,

2. $T(ab) = T(a)T(b)$,

3. $T(\alpha a) = \alpha T(a)$, *for all $a, b \in A$ and $\alpha \in \mathbb{F}$.*

We continue with the definition of a module over an algebra over a field $\mathbb{F}$.

Definition 1.1.7. *Let $\mathbb{F}$ be a field and A an algebra over $\mathbb{F}$. We say that M is a* ***right A-module****, denoted by M_A, if it is an $\mathbb{F}$-vector space together with a mapping*

$$\cdot : M \times A \to M, \ \ (m, a) \mapsto ma,$$

satisfying the following axioms:

1. $m \cdot (a_1 a_2) = (m \cdot a_1) a_2$,
2. $m \cdot (a_1 + a_2) = (m \cdot a_1) + (m \cdot a_2)$,
3. $(m_1 + m_2) \cdot a = (m_1 \cdot a) + (m_2 \cdot a)$,
4. $\lambda(m \cdot a) = (\lambda m) \cdot a$,

for all $m, m_1, m_2 \in M$, $a, a_1, a_2 \in A$ and $\lambda \in \mathbb{F}$. A left A-module is defined similarly. Let B be an algebra over $\mathbb{F}$. M is called an ***A-B-bimodule*** *if M is both a left A-module and a right B-module such that $(am)b = a(mb)$ and $(\alpha 1_A)a = a(\alpha 1_B)$ for all $a \in A$ and $b \in B$, $m \in M$ and $\alpha \in \mathbb{F}$. It will be denoted by ${}_A M_B$.*

Throughout we will assume that all of our modules are finite dimensional over an algebra.

An example of a right A-module that occurs quite often in representation theory is the following:

Example 1.1.8. *Let $\mathbb{F}$ be a field and A an algebra over $\mathbb{F}$. A is a right A-module with the natural right multiplication in A. This module will be denoted by A_A and is called the* ***right regular A-module****. The left regular A-module is defined similarly.*

Definition 1.1.9. *Let A be an algebra over a field $\mathbb{F}$. Let M be an A-module and N be an $\mathbb{F}$-vector subspace of M. Then N is an* ***A-submodule*** *of M if $na \in N$ for all*

$a \in A$ and $n \in N$. An A-submodule N of M is called a nontrivial submodule if N is neither $\{0\}$ nor M. Note that the A-submodules of the right regular module A_A are the right ideals of A.

We give a special name to modules that have only trivial submodules.

Definition 1.1.10. *Let A be an algebra over a field $\mathbb{F}$. An A-module M is called **simple**, or irreducible, if the only A-submodules of M are 0 and M.*

We proceed by introducing some notation which will be used throughout this dissertation.

Let $\mathbb{F}$ be a field. If V is an n-dimensional vector space over $\mathbb{F}$, we write $GL(V)$ for the group of all invertible $\mathbb{F}$-linear transformations of V. Let $T \in GL(V)$ and let $B = \{v_1, v_2, \ldots, v_n\}$ be a basis of V. For each $i \in \{1, 2, \ldots, n\}$, we write the image of v_i under T in terms of the basis B:

$$(v_i)T = \sum_{j=1}^{n} \alpha_{ij} v_j,$$

where $(v_i)T$ is the image of v_i under the invertible linear transformation T.

Let $M_B(T) = (\alpha_{ij})$ be the $n \times n$ matrix whose (i, j) entry is α_{ij}, i.e. use the coefficients of the v_j's in the above computation of $(v_i)T$ for the i^{th} row of this matrix. The matrix $M_B(T)$ is called the **matrix of T with respect to the basis B**. The mapping $T \overset{F}{\mapsto} M_B(T)$ maps $GL(V)$ onto the invertible $n \times n$ matrices, which we denote by $GL(n, \mathbb{F})$.

Definition 1.1.11. *Let G be a group and V a finite dimensional vector space over a field $\mathbb{F}$. An **$\mathbb{F}$-representation** of a group G is a homomorphism T from G to $GL(V)$. If a basis B for V is chosen then the composite map*

$$G \xrightarrow{T} GL(V) \xrightarrow{F} GL(n, \mathbb{F})$$

is called an $\mathbb{F}$***-matrix representation*** *of* G *and it will be denoted by* $\hat{T}$*. The* $\mathbb{F}$*-dimension of* V *is called the degree of* T*. If* T *(respectively* $\hat{T}$*) is injective, then* T *(respectively* $\hat{T}$*) is called* ***faithful****.*

Example 1.1.12. *Let* $\mathbb{F}$ *be a field. If* V *is an* $\mathbb{F}$*-vector space we get a representation* $T\colon G \to GL(V)$*,* $g \mapsto id_V$ *for a group* G*. This representation is called the* ***trivial representation*** *of* G*.*

Example 1.1.13. *Let* G *be a finite group,* Ω *be a finite* G*-set, and* $\mathbb{F}$ *be a field. Let* V *be an* $\mathbb{F}$*-vector space with basis* $\{m_w\colon w \in \Omega\}$*, where the basis elements correspond bijectively to the elements of* Ω*. Each element* $x \in G$ *defines an* $\mathbb{F}$*-linear map* $T(x)\colon V \to V$*, where* $m_w T(x) = m_{xw}$*, for* $w \in \Omega$*. Thus* $T\colon G \to GL(V)$ *is a representation of* G*, called a permutation representation of* G*.*

Example 1.1.14. *Let* H *be a subgroup of a finite group* G *and* $\mathbb{F}$ *a field. Let* Ω *be the* G*-set of right cosets* $\{Hg_i\}$ *of* H*. Let* V *be an* $\mathbb{F}$*-vector space with basis* $\{v_i\}$ *corresponding to the right cosets* $\{Hg_i\}$*. The action of* G *on the basis defined by*

$$v_i g = v_j \text{ if } Hg_i g = Hg_j,\ 1 \leq i \leq n,$$

defines a permutation representation of G*, called the permutation representation of* G *on the cosets of* H*.*

Definition 1.1.15. *Let* $\mathbb{F}$ *be a field,* V *and* W *be* $\mathbb{F}$*-vector spaces. Two representations* $T\colon G \to GL(V)$ *and* $T'\colon G \to GL(W)$ *are called* ***equivalent*** *if there is an* $\mathbb{F}$*-vector space isomorphism* $\rho\colon V \to W$ *such that*

$$T'(g) = \rho T(g) \rho^{-1} \text{ for all } g \in G.$$

Definition 1.1.16. *Let* G *be a group and* $\mathbb{F}$ *be a field and let* V *be an* $\mathbb{F}$*-vector space. If* $T\colon G \to GL(V)$ *is a representation of* G *then a subspace* W *of* V *is called* T***-invariant*** *if* $WT(x) \subseteq W$ *for all* $x \in G$*. If* V *has a proper nonzero* T*-invariant* *subspace W then T is called* ***reducible****, otherwise T is called* ***irreducible****, or simple.*

In the next chapter we will present some algorithms which determine whether a given representation is irreducible.

Definition 1.1.17. *Let $\mathbb{F}$ be a field and V be an $\mathbb{F}$-vector space. If $T\colon G \to GL(V)$ is a representation of G of degree n, define the* ***centralizer*** *of T to be the set of all linear transformations $H\colon V \to V$ for which $HT(x) = T(x)H$ for all $x \in G$. We write $C(T)$ for the centralizer of T. The centralizer $C(\hat{T})$ of a corresponding matrix representation $\hat{T}$ is the set of $n \times n$ matrices over $\mathbb{F}$ that commute with all $\hat{T}(x)$, for all $x \in G$.*

Definition 1.1.18. *Let A be a finite dimensional $\mathbb{F}$-algebra and V a finite dimensional $\mathbb{F}$-vector space. An* ***$\mathbb{F}$-representation of*** *A is an $\mathbb{F}$-algebra homomorphism*

$$T\colon A \to \mathrm{End}_{\mathbb{F}}(V).$$

Definition 1.1.19. *Let $\mathbb{F}$ be a field, V be an $\mathbb{F}$-vector space and G be a finite group. Any representation $T\colon G \to GL(V)$ extends by $\mathbb{F}$-linearity to an $\mathbb{F}$-algebra homomorphism $T\colon \mathbb{F}G \to \mathrm{End}_{\mathbb{F}} V$ which is called a* ***representation of the group algebra****.*

Using the notation from the above definition, V becomes a right $\mathbb{F}G$-module via $v \cdot a = vT(a)$. Conversely, if V is any finite dimensional $\mathbb{F}G$-module, then we obtain a representation $T\colon \mathbb{F}G \to \mathrm{End}_{\mathbb{F}} V$ of the group algebra $\mathbb{F}G$ by defining

$$vT(a) := v \cdot a.$$

By restricting T to G we obtain a representation of G.

Example 1.1.20. *For a finite group G and a field $\mathbb{F}$ the right regular $\mathbb{F}G$-module $\mathbb{F}G_{\mathbb{F}G}$ corresponds to the right regular representation of G, see Example* 1.1.14.

Definition 1.1.21. *Let A be an algebra over a field $\mathbb{F}$. A descending chain $M_1 > M_2 > M_3 \cdots$ of submodules in an A-module is said to terminate if there is an integer k such that $M_i = M_k$ for all $i \geq k$. An A-module for which every descending chain of submodules terminates, is called **Artinian**.*

*Similarly, an ascending chain $M_1 < M_2 < M_3 \ldots$ of submodules in an A-module is said to terminate if there is an integer n such that $M_i = M_n$ for all $i \geq n$. An A-module for which every ascending chain of submodules terminates, is called **Noetherian**.*

Definition 1.1.22. *Let A be an algebra over a field $\mathbb{F}$ and M be an A-module. An ascending chain*

$$0 = M_0 < M_1 < \cdots < M_n = M$$

*of A-submodules of M is called a **composition series** for M if M_i/M_{i-1} is a simple A-module for $1 \leq i \leq n$. We call the natural number n the length of the composition series. The factors M_i/M_{i-1}, for $1 \leq i \leq n$, are called the **composition factors** of M.*

The composition factors of an A-module are uniquely determined by the following theorem.

Theorem 1.1.23. *(Jordan-Hölder) Let A be an algebra over a field $\mathbb{F}$. Suppose M is an A-module having two composition series:*

$$0 = M_0 < M_1 < \cdots M_m = M$$

and

$$0 = N_0 < N_1 < \cdots N_n = M$$

Then $m = n$ and there exists a permutation π of the numbers $1, \ldots, m$ such that

$$M_i/M_{i-1} \cong N_{\pi(i)}/N_{\pi(i)-1}.$$

Proof. A proof of the Jordan-Hölder theorem is found in [11, page 62]. □

Lemma 1.1.24. *Let $\mathbb{F}$ be a field and A an algebra over $\mathbb{F}$. An A-module M has a composition series if and only if M is both Artinian and Noetherian.*

Proof. For a proof, see [4, page 41]. □

Definition 1.1.25. *Let A be an algebra over a field $\mathbb{F}$. Let M and N be A-modules. Then $\rho\colon M \to N$ is an A-**homomorphism** if $\rho(m_1 \cdot a + m_2) = \rho(m_1) \cdot a + \rho(m_2)$ for all $m_1, m_2 \in M$ and $a \in A$. If ρ is a bijection then it is called an A-**isomorphism**. We use $\mathrm{Hom}_A(M, N)$ to denote the $\mathbb{F}$-vector space of all A-homomorphisms from M to N and $\mathrm{End}_A(M)$ for $\mathrm{Hom}_A(M, M)$. We denote by $\mathrm{Ker}(\rho)$ and $\mathrm{Im}(\rho)$ the kernel and image of the A-homomorphism ρ respectively.*

One of the fundamental observations in representation theory is known as:

Lemma 1.1.26. ***(Schur's Lemma:)*** *Let A be an algebra over a field $\mathbb{F}$ and M, N be simple A-modules. Then $\mathrm{Hom}_A(M, N) = 0$ if M is not isomorphic to N and $\mathrm{End}_A(M)$ is a division ring.*

Proof. A proof of this can be found in [5, page 36]. □

Definition 1.1.27. *If A is an algebra over a field $\mathbb{F}$ and $\mathbb{L}$ is an extension field of $\mathbb{F}$ one can form the algebra $\mathbb{L}A := \mathbb{L} \otimes_{\mathbb{F}} A$ over $\mathbb{L}$. Multiplication in $\mathbb{L}A$ is defined by*

$$(\lambda \otimes a) \cdot (\mu \otimes b) := \lambda\mu \otimes ab$$

for all $\lambda, \mu \in \mathbb{L}$ and $a, b \in A$, and scalar multiplication is defined by

$$\lambda \cdot (\mu \otimes a) := (\lambda\mu \otimes a).$$

Then for any A-module V the tensor product $\mathbb{L} \otimes_{\mathbb{F}} V$ becomes an $\mathbb{L}A$-module via

$$(\lambda \otimes m) \cdot (\mu \otimes a) := \lambda\mu \otimes (ma)$$

for all $\lambda, \mu \in \mathbb{L}$, $m \in M$, and $a \in A$. The A-module V is called ***absolutely simple*** *if $\mathbb{L}V$ is simple for any field extension $\mathbb{L}$ of K. A field $\mathbb{L}$ such that the composition factors of $\mathbb{L}V$ are absolutely irreducible is called a* ***splitting field*** *for V. A field $\mathbb{L}$ is called a splitting field for A if $\mathbb{L}$ is a splitting field for every simple A-module. If G is a finite group and $\mathbb{L}$ is a splitting field for the group algebra $\mathbb{L}G$, then $\mathbb{L}$ is also called a splitting field for G.*

1.2 Structure theorems

Definition 1.2.1. *Let A be an algebra over a field $\mathbb{F}$. An A-module M is said to be an* ***internal direct sum*** *of $M_1 \oplus M_2 \oplus \cdots \oplus M_r$ of submodules $M_1, M_2, \ldots M_r$ of M for r a positive integer if the following conditions are satisfied:*

1. $M = M_1 + M_2 + \cdots M_r$
2. $M_i \cap (M_1 + \cdots + M_{i-1} + M_{i+1} + \cdots + M_r) = 0$ *for* $i = 1, \ldots, r$.

$M_1 \oplus \cdots \oplus M_r$ is said to be a ***decomposition*** *of M. The submodules M_i are called* ***direct summands*** *of M. In the case $r > 1$ and $M_i \neq \{0\}$ for all $i \in \{1, \ldots, r\}$, we call M* ***decomposable***. *If M cannot be written as a direct sum of two nontrivial submodules then M is called* ***indecomposable***.

Definition 1.2.2. *Let A be an algebra over a field $\mathbb{F}$. An A-module M is called* ***semisimple*** *if it is a direct sum of simple modules, i.e. $M = M_1 \oplus M_2 \oplus \cdots \oplus M_r$ with each M_i a simple A-submodule of M for $1 \leq i \leq r$. An algebra A over $\mathbb{F}$ is called* ***semisimple*** *if the right regular A-module, A_A, is semisimple.*

The following theorem characterizes semisimple group algebras.

Theorem 1.2.3. *(Maschke's Theorem) Let G be a group and $\mathbb{F}$ be a field. The group algebra $\mathbb{F}G$ is semisimple if and only if the characteristic of $\mathbb{F}$ does not divide the order of G.*

Proof. For a proof see [11, page 194]. □

Lemma 1.2.4. *Let A be an algebra over a field $\mathbb{F}$, M be a finite dimensional A-module and $\rho \in \operatorname{End}_A(M)$. Then*

$$M = \operatorname{Im} \rho^m \oplus \operatorname{Ker} \rho^m$$

for m sufficiently large.

Proof. See for example [5, page 34]. □

In representation theory and in the study of A-modules the following A-submodules are important.

Definition 1.2.5. *Let A be an algebra over a field $\mathbb{F}$. If M is an A-module, then the **radical** of M is defined as the intersection of all maximal submodules of M, i.e.*

$$\operatorname{Rad}(M) := \cap\{U \leq_A M | M/U \,\text{is simple}\}$$

*The **socle** of an A-module M is the sum of all the simple submodules of M, i.e.*

$$\operatorname{Soc}(M) := \sum\{U \leq_A M | U \,\text{is simple}\}$$

Using the definition of socle of an A-module M, we can define the following series of submodules of M.

Definition 1.2.6. *Let A be a finite dimensional algebra over a field $\mathbb{F}$. Given a finite dimensional A-module M we define the socle series of M as follows: The first member $\operatorname{Soc}_1(M) = \operatorname{Soc}(M)$ and for $i > 1$ we define the ith member $\operatorname{Soc}_i(M)$ by $\operatorname{Soc}_i(M)/\operatorname{Soc}_{i-1}(M) = \operatorname{Soc}(M/\operatorname{Soc}_{i-1}(M))$. The successive quotients are called the **socle layers** of M. Since M is finite dimensional there is a smallest $l \in \mathbb{N}$ such that $\operatorname{Soc}_l(M) = M$ which is called the **socle length** of M.*

A central type of module that we need to discuss is a generalization of a free module.

Definition 1.2.7. *Let A be an algebra over a field $\mathbb{F}$. An A-module V is called projective if V is a direct summand of a free A-module. An A-module V is called* ***projective indecomposable****, often abbreviated as* ***P.I.M.****, if V is projective and indecomposable.*

For a finite dimensional algebra A over a field $\mathbb{F}$ there is a one to one correspondence between the simple A-modules and the projective indecomposable A-modules which is described in the following theorem.

Theorem 1.2.8. *Let A be a finite dimensional algebra over $\mathbb{F}$. Let $S_1, \ldots, S_r$ be the simple A-modules up to isomorphism and $D_i = \operatorname{End}_A S_i$. Then there are r projective indecomposable modules $P_1, \ldots, P_r$ up to isomorphism with $P_i / \operatorname{Rad} P_i \cong S_i$ and*

$$A_A = \oplus_{i=1}^{r} (P_{i,1} \oplus \cdots \oplus P_{i,f_i}) \text{ with } P_{i,j} \cong P_i$$

where $f_i = \dim_{D_i} S_i$. If $\mathbb{F}$ is a splitting field for A then the f_i are just the degrees of the irreducible representations of A.

Proof. A proof of this theorem can be found in [2, page 14]. □

Definition 1.2.9. *Let A be an algebra over a field $\mathbb{F}$. If V is an A-module then a projective A-module P is called a* ***projective cover*** *of V if $P / \operatorname{Rad} P \cong V / \operatorname{Rad} V$ and P is unique up to isomorphism. Note that if V is a simple A-module then a projective A-module P is the projective cover of V if $P / \operatorname{Rad} P \cong V$. The projective covers of V are unique up to isomorphism. With the notation used in Theorem 1.2.8 each P_i is the projective cover of the simple A-module S_i for $1 \leq i \leq r$.*

Definition 1.2.10. *Let $\mathbb{F}$ be a field, H a subgroup of a group G, and V an $\mathbb{F}G$-module. The* ***restriction of** V **to** H*, V_H, *is V regarded as an $\mathbb{F}H$-module.*

Definition 1.2.11. *Let $\mathbb{F}$ be a field, H a subgroup of a group G, and W an $\mathbb{F}H$-module. Then the* ***induction of*** *W* ***to*** *G, denoted by W^G, is given by $W^G = W_{\mathbb{F}H} \otimes \mathbb{F}G$, where $\mathbb{F}G$ is considered as an $\mathbb{F}H$-$\mathbb{F}G$-bimodule.*

Example 1.2.12. *Let H be a subgroup of a finite group G and $\mathbb{F}$ a field. The induction of the trivial $\mathbb{F}H$-module 1_H to G, denoted by 1_H^G, is isomorphic to the permutation module of G on the cosets of H. For the details, see [4, page 231].*

Theorem 1.2.13. *Let H be a subgroup of G. If W is a projective $\mathbb{F}H$-module then W^G is a projective $\mathbb{F}G$-module.*

Proof. For a proof see [13, page 188]. □

Lemma 1.2.14. *Let $\mathbb{F}$ be a field whose characteristic does not divide $|G|$. Then every finitely generated $\mathbb{F}G$-module is projective.*

Proof. This is a consequence of Maschke's theorem, see Theorem 1.2.3. □

1.3 Idempotents

Certain elements of an algebra are essential in the study of representation theory of a finite dimensional algebra.

Definition 1.3.1. *Let A be an algebra over a field $\mathbb{F}$. A nonzero element e of A is called an* ***idempotent*** *if $e^2 = e$. Two idempotents e_1 and e_2 in A are called* ***orthogonal*** *if $e_1e_2 = e_2e_1 = 0$. An idempotent $e \in A$ is called* ***primitive*** *if it can not be written as the sum of two orthogonal idempotents.*

As described in the following proposition primitive idempotents of an algebra A correspond with indecomposable right ideals of A.

Proposition 1.3.2. *Let A be an algebra over a field $\mathbb{F}$. An idempotent $e \in A$ is primitive if and only if eA is an indecomposable right ideal of A.*

Proof. See [4, page 119]. □

By Theorem 1.2.8 we know that there is a one to one correspondence between the projective indecomposable A-modules and the simple A-modules for a finite dimensional algebra A over $\mathbb{F}$. The next theorem gives an explicit description of the P.I.M.s.

Theorem 1.3.3. *If A is a finite dimensional algebra over $\mathbb{F}$, then P is a projective indecomposable A-module if and only if $P \cong eA$ for some primitive idempotent $e \in A$.*

Proof. For a proof see [4, page 120]. □

The following theorem shows that the module homomorphisms from a P.I.M. eA to an A-module V are determined by the image of the idempotent e.

Theorem 1.3.4. *Let A be an algebra over a field $\mathbb{F}$. If $e \in A$ is an idempotent and V an A-module, then*

1. $\mathrm{Hom}_A(eA, V) \cong Ve$ *as $\mathbb{F}$ vector spaces, and*
2. $\mathrm{End}_A(eA) \cong (eAe)^{op}$ *as rings.*

Proof. For a proof see [4, page 45]. □

1.4 Character Theory

The purpose of this section is to introduce the concepts of characters and Brauer characters of finite groups. This material is well known and can be found for example in [4]. Throughout this section we assume that A is a finite dimensional algebra over a field $\mathbb{F}$. In this dissertation we are interested in characters of a finite group G that take values in a field of characteristic zero. So throughout this section we assume that $\mathbb{F}$ is a splitting field for G of characteristic 0.

Definition 1.4.1. *Let V be an A-module and let $\rho\colon A \to \operatorname{End}_{\mathbb{F}} V$ be the corresponding representation. Then the function*

$$\chi_V = \chi_\rho\colon A \to \mathbb{F},\ \ a \mapsto \operatorname{trace}\rho(a)$$

is called the ***character*** *of V or of ρ. A character of an irreducible representation is called* ***irreducible****. If A is the group algebra $\mathbb{F}G$ over a field $\mathbb{F}$ then $\chi = \chi_V$ is often identified with its restriction*

$$\chi|_G\colon G \to \mathbb{F},\ \ g \to \operatorname{trace}(\rho(g)).$$

A character of a group G over $\mathbb{F}$ is a character of a representation of G over $\mathbb{F}$. The set of irreducible characters of G over $\mathbb{F}$ will be denoted by $\operatorname{Irr}_{\mathbb{F}}(G)$.

Example 1.4.2. *Let G be a group. The character of the trivial representation of G (see page 12) of degree one is called the trivial character and will be denoted by 1_G, thus $1_G(g) = 1$ for all $g \in G$.*

Another important example is the following:

Example 1.4.3. *The character of the regular representation ρ of a group G (see page 12) over $\mathbb{F}$ is the* ***regular character*** *ρ_G given by*

$$\rho_G(g) = \begin{cases} 0 & \textit{for} \quad 1 \neq g \in G \\ |G| \cdot 1_{\mathbb{F}} & \textit{for} \quad g = 1 \end{cases}$$

Permutation representations and their characters are essential for algorithmic methods used in representation theory.

Example 1.4.4. *Suppose G acts on a set Ω and δ is the corresponding permutation representation of G over $\mathbb{F}$ (see page 12), then we have $\chi_\delta(g) = |Fix_\Omega(g)| \cdot 1_{\mathbb{F}}$, where $Fix_\Omega(g) = \{x \in \Omega | gx = x\}$ is the set of fixed points of g on Ω. Such a character is called a* ***permutation character****.*

In the following we give some basic properties of characters. The second statement of the following proposition states that the characters of G are constant on conjugacy classes of G.

Proposition 1.4.5. *Let χ_ρ be the character afforded by a representation ρ of G of degree n. Then the following hold.*

1. $\chi_\rho(1) = n$,
2. $\chi_\rho(tst^{-1}) = \chi_\rho(s)$, *for* $s, t \in G$.

Proof. For a proof see [4, page 209] □

The following is a fundamental result in the ordinary character theory.

Proposition 1.4.6. *Let G be a group and $\mathbb{F}$ a splitting field of G with characteristic 0. Then $|\operatorname{Irr}_{\mathbb{F}}(G)|$ equals the number of conjugacy classes of G.*

Proof. For a proof see [8, page 36]. □

Definition 1.4.7. *A class function on a group G is a function $\rho\colon G \to \mathbb{F}$ that is constant on the conjugacy classes of G.*

$$\operatorname{cf}(G, \mathbb{F}) := \{\psi\colon G \to \mathbb{F} | \psi(h^{-1}gh) = \psi(g) \text{ for all } g, h \in G\}$$

*is called the **space of $\mathbb{F}$-class functions** on G. We define a symmetric bilinear form on $\operatorname{cf}(G, \mathbb{F})$ in the following way: For $\psi, \rho \in \operatorname{cf}(G, \mathbb{F})$ we put*

$$(\rho, \psi)_G := \frac{1}{|G|} \sum_{g \in G} \rho(g)\psi(g^{-1}).$$

*This is called the **scalar product** of two characters.*

Theorem 1.4.8. *(Orthogonality Relations) Let G be a finite group and $\mathbb{F}$ a splitting field of G of characteristic 0. Set $Irr_{\mathbb{F}}(G) := \{\chi_1, \ldots, \chi_r\}$ and let $\{g_1, \ldots, g_r\}$ be representatives of the conjugacy classes of G. Then $Irr_{\mathbb{F}}(G)$ is an orthonormal basis of* $\mathrm{cf}(G, \mathbb{F})$ *and*

$$(\chi_i, \chi_j)_G = \sum_{g \in G} \chi_i(g_k)\chi_j(g_k^{-1}) = \sum_{k=1}^{r} \frac{1}{|C_G(g_k)|} \chi_i(g_k)\chi_j(g_k^{-1}) = \delta_{ij},$$

$$\sum_{k=1}^{r} \chi_i(g_k)\chi_j(g_k^{-1}) = |C_G(g_k)|\delta_{ij},$$

where $C_G(g_k)$ denotes the centralizer of g_k, i.e. $C_G(g_k) = \{x \in G | xg_k = g_k x\}$.

Proof. For a proof see [4, page 213]. □

Proposition 1.4.9. *Let G be a finite group, $\mathbb{F}$ be a splitting field of G of characteristic 0 and $\mathrm{Irr}_{\mathbb{F}}\, G = \{\chi_1, \ldots, \chi_r\}$. If V is an arbitrary $\mathbb{F}G$-module with character χ_V, then*

$$\chi_V = \sum_{i=1}^{r} (\chi_V, \chi_i)_G \chi_i.$$

Definition 1.4.10. *Let $\mathbb{F}$ be a field, H a subgroup of a group G, and V an $\mathbb{F}G$-module. Let V_H be the restriction of V to H (see page 19). If χ is the character of V then the character of V_H is simply the character χ restricted to H and will be denoted by $\chi|_H$.*

Definition 1.4.11. *Let $\mathbb{F}$ be a field, H a subgroup of a group G, and W an $\mathbb{F}H$-module. Let W^G be the induction of W to G (see page 19). If χ is a character of H then the character χ^G of W^G is given by*

$$\chi^G(g) = \frac{1}{|H|} \sum_{x \in G} \chi^o(xgx^{-1}),$$

where

$$\chi^o(y) = \begin{cases} \chi(y) \text{ if } y \in G \\ 0 \text{ otherwise} \end{cases}$$

is called the ***induced character****.*

1.5 Brauer Characters

Definition 1.5.1. *Let G be a finite group and let p be a prime number. An element $x \in G$ is called p-regular if its order is not divisible by p, and is called p-singular otherwise. We use $G_{p'}$ for the set of all p-regular elements of G.*

Definition 1.5.2. *Let G be a finite group. The* ***exponent*** *of G, denoted by* $\exp(G)$ *is the smallest integer n such that $g^n = 1$ for all $g \in G$.*

For a prime number p, let

$$U_{p'} := \{\zeta \in \mathbb{C} | \zeta^m = 1 \ \ for \ \ some \ \ m \ \ with \ \ p \nmid m\}$$

be the multiplicative group of roots of unity of order coprime to p and we define

$$U_{p',m} := \{\zeta \in U_{p'} | \zeta^m = 1\}.$$

Definition 1.5.3. *Let G be a finite group of exponent $m = p^r q$ with $p \nmid q$ and $\mathbb{K}$ be a field of characteristic $p > 0$ containing the q-th roots of unity. Assume that W is an $\mathbb{K}G$-module with $\dim_{\mathbb{K}} W = n$ and with representation $\delta \colon G \to GL(W)$. Let $\theta \colon \mathbb{Z}[\zeta_m] \to \mathbb{K}$ be a fixed ring homomorphism. For $g \in G_{p'}$ the eigenvalues of $\delta(g)$ are m-th roots of unity in $\mathbb{F}$ and hence are of the form $\theta(\zeta_1(g)), \dots, \theta(\zeta_n(g))$ with uniquely determined $\zeta_1(g), \dots \zeta_n(g) \in U_{p',m}$. We then define*

$$\rho_W(g) := \zeta_1(g) + \cdots + \zeta_n(g)$$

and call

$$\rho_W \colon G_{p'} \to \mathbb{C}, \ \ g \to \zeta_1(g) + \cdots + \zeta_n(g)$$

the ***Brauer character*** *of G afforded by W or δ with respect to θ. If δ is an irreducible representation of G, then the Brauer character ρ afforded by δ is called irreducible. The set of irreducible Brauer characters of G over a splitting field of characteristic p will be denoted by* $\mathrm{IBr}_p(G)$.

Obviously this definition of a Brauer character depends on θ. In order to avoid this ambiguity a definite choice for θ is made. The details of this can be found in [4, page 420].

We give some basic consequences of the definition.

Lemma 1.5.4. *Let G be a finite group, $\mathbb{F}$ be a splitting field of G of characteristic 0 and $\mathbb{K}$ be a splitting field of G with characteristic $p > 0$. Let ρ be a Brauer character of G afforded by the $\mathbb{K}G$-module W. Then*

1. *ρ is a class function on $G_{p'}$, i.e. $\rho \in \mathrm{cf}(G_{p'}, \mathbb{C})$.*
2. *If $p \nmid |G|$ then $\mathrm{Irr}_{\mathbb{F}}(G) = \mathrm{IBr}_p(G)$.*
3. *If $H \leq G$ is a subgroup of G, $p \nmid |H|$ and ρ is a Brauer character of G then $\rho|_H$ is an ordinary character of H.*

Proof. For a proof see [13, pages 299,300]. □

Theorem 1.5.5. *Let χ be an ordinary character of a group G and let $\chi|_{G_{p'}}$ denote the restriction of χ to the p-regular elements of G. Then $\chi|_{G_{p'}}$ is a Brauer character of G.*

Proof. A proof of this theorem can be found in [9, page 266]. □

In the following we describe how to decompose the Brauer characters that come from restricting irreducible characters.

Definition 1.5.6. *Let G be a finite group, $\mathbb{F}$ be a splitting field of G of characteristic 0 and $\mathbb{K}$ be a splitting field of G with characteristic $p > 0$. Let $\mathrm{Irr}_{\mathbb{F}}(G) = \{\chi_1, \cdots, \chi_k\}$ and $\mathrm{IBr}_p(G) = \{\varphi_1, \cdots, \varphi_l\}$. We write*

$$\chi_i|_{G_{p'}} = \sum_{j=1}^{l} d_{\chi_i,\varphi_j} \varphi_j.$$

*The nonnegative integers d_{χ_i,φ_j} are uniquely defined and are called the p-**decomposition numbers** of G. The matrix $D = [d_{\chi_i,\varphi_j}]$ is called the **p-decomposition matrix** of G.*

In practice the irreducible Brauer characters, $\mathrm{IBr}_p(G) = \{\varphi_1, \dots, \varphi_l\}$ will be given by a matrix $[\varphi_i(g_j)]_{1\leq i\leq l, 1\leq j\leq s}$, where $\{g_1, \dots, g_s\}$ are representatives of the p-regular conjugacy classes of G. This matrix is called the **p-Brauer character table** of G. As stated in the next theorem this is a square matrix.

Theorem 1.5.7. *The number $|\,\mathrm{IBr}_p(G)|$ of irreducible Brauer characters of G equals the number of p-regular conjugacy classes of G.*

Proof. For a proof see [13, page 301]. □

The decomposition numbers we defined above are essential for the connection of the theory of ordinary characters with the theory of Brauer characters.

Definition 1.5.8. *Let G be a finite group, $\mathbb{F}$ be a splitting field of G with characteristic 0 and $\mathbb{K}$ be a splitting field of G with characteristic $p > 0$. For $\varphi \in \mathrm{IBr}_p(G)$ the ordinary character*

$$\Phi_\varphi = \sum_{\chi\in Irr_{\mathbb{F}}(G)} d_{\chi,\varphi}\chi$$

*is called the **projective indecomposable character** associated to φ. The character $\Phi_\varphi|_{G_{p'}}$ restricted to the p-regular classes of G is the Brauer character of the projective cover $P(S)$ of a simple $\mathbb{K}G$-module S with Brauer character φ.*

Theorem 1.5.9. *The projective indecomposable characters Φ_φ for $\varphi \in IBr(G)$ form a basis for the class functions on G which are zero on the set of all elements of $G\backslash G_{p'}$.*

Proof. A proof of this theorem can be found in [13, page 307]. □

Chapter 2

DIRECT METHODS

2.1 The Norton Criterion

Throughout this chapter $\mathbb{F}$ denotes a finite field of characteristic p, p a prime, and A a finite dimensional algebra over $\mathbb{F}$. The basic algorithmic problems we plan to study in this chapter are the following:

1. Decide whether an A-module is simple, i.e. whether an A-representation is irreducible.

2. Find a nontrivial A-submodule of a reducible A-module.

We will see that there are algorithmic answers to both questions. We introduce the Norton Criterion, which is the essential tool for proving the irreducibility of a given A-module.

Remark 2.1.1. *If the efficiency of such an algorithm does not concern us, then there is an obvious way of proving the irreducibility of an A-module V. We can run through all nonzero vectors $v \in V$, determine a basis for vA, where vA is the smallest A-submodule containing v and check whether vA is a proper submodule of V. However this method depends heavily on the field size and the dimension of V. We will see that this is essentially unnecessary.*

The algorithm which determines a basis for vA is called the **Spinning Algorithm**. Since it is an important part of the Norton Criterion we want to describe how the algorithm works.

The Spinning Algorithm:

Input: A finite dimensional A-module V, the corresponding matrix representation δ of A in terms of matrices $\{A_1 = \delta(a_1), \ldots, A_k = \delta(a_k)\}$ for an algebra generating set $\{a_1, \ldots, a_k\}$ of A, and a nonzero $v \in V$.

Calculation:

Set $B = \{v\}$

+ For all $b \in B$ do

  - For all $A_i \in \{A_1, \ldots, A_k\}$ do

    * If bA_i is not in the $\text{span}_{\mathbb{F}}(B)$ then add bA_i to B.

    * End if

  - Next A_i

+ Next b

Output: A basis B for the A-submodule vA.

From the algorithm, we see that since V is a finite dimensional A-module, a basis B for vA can be obtained in finitely many steps.

Let us recall some basic definitions before we state the Norton criterion.

Definition 2.1.2. *Let V be a finite dimensional $\mathbb{F}$-vector space. Let $\mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$ be the **dual space** denoted by V^*. If V is a right A-module for the $\mathbb{F}$-algebra A then V^* is a left A-module in the following way: For $\lambda \in V^*$ and $a \in A$ we define $a \cdot \lambda(v) := \lambda(v \cdot a)$ for all $v \in V$. If W is an $\mathbb{F}$-subspace of V, then we define the **annihilator** W^0 of W in V^* to be $W^0 = \{\lambda \in V^* | \lambda(w) = 0\}$ for all $w \in W$. Note that if W is a right A-submodule of V, then W^0 is a left A-submodule of V^*.*

In the Norton Criterion we will use the following facts about dual vector spaces.

Theorem 2.1.3. *Let V be an A- module and let W be an A-submodule of V. Then $V^*/W^0 \cong W^*$ as left A-modules and $(V^*)^* \cong V$ as right A-modules. If M is an $\mathbb{F}$-vector space and $\phi \in \mathrm{Hom}_{\mathbb{F}}(V, M)$, then we denote by $\phi^* \in \mathrm{Hom}_{\mathbb{F}}(M^*, V^*)$ the linear* *map defined by* $\phi^*(\lambda)(v) = \lambda(\phi(v))$ *for all* $\lambda \in M^*$ *and* $v \in V$*. The kernel of* ϕ^* *is given by the annihilator of the image of* ϕ*.*

Proof. See for example [11, page 73].

□

The following theorem gives necessary and sufficient conditions for a representation of a finite dimensional algebra to be irreducible. This was first implemented by R. Parker see [17].

Theorem 2.1.4. ***(The Norton Criterion)*** *Let* A *be an algebra over a field* $\mathbb{F}$*. Let* V *be a finite dimensional* A*-module and let* $\delta : A \to \mathrm{End}_{\mathbb{F}}(V)$ *be the corresponding representation of* A*. Moreover let* $a \in A$ *with* $0 < \mathrm{Ker}(\delta(a)) < V$*. Then* δ *is an irreducible representation of* A *if and only if*

1. *For all nonzero elements* $v \in \mathrm{Ker}(\delta(a))$ *it holds that* $vA = V$*,*

2. *There exists a nonzero element* $v^* \in \mathrm{Ker}(\delta^*(a))$ *with* $Av^* = V^*$*.*

Proof. For a proof see [12, page 28].

□

The Norton criterion immediately leads to an algorithm that enables us to verify that a given representation of an algebra A over $\mathbb{F}$ is irreducible.

Algorithm 2.1.5. ***The Norton Criterion***

Input: *We start with a matrix representation* δ *of* A *in terms of matrices* $\{A_1 = \delta(a_1), \ldots, A_k = \delta(a_k)\}$ *for a generating set* $\{a_1, \ldots, a_k\}$ *of* A*,*

1. *Pick an element* a *in* A*,*

2. *Determine* $\mathrm{Ker}(\delta(a))$*,*

3. *If* $\mathrm{Ker}(\delta(a))$ *is nonzero then for all nonzero vectors* $v \in \mathrm{Ker}(\delta(a))$*, up to scalars, determine a basis of* vA *by using the spinning algorithm. If* $vA < V$ *then return the matrices with respect to this basis of* vA *for the generating system on* vA*.*

4. *If* $vA = V$ *for all nonzero* $v \in \mathrm{Ker}(\delta(a))$*, then determine one vector* v^* *such that* $\delta(a)^* v^* = 0$ *and determine a basis of* Av^* *by using the spinning algorithm with transposed input, namely* $\{A_1^T, \ldots, A_k^T\}$*. If* $Av^* < V^*$*, then return the transposed matrices with respect to this basis for the generating system on* Av^**. If* $Av^* = V^*$*, then return the answer 'irreducible'.*

Output: *Either the information that* V *is irreducible, or a matrix representation of* A *on an* A*-submodule* W*.*

Note that in this algorithm only the basic matrix operations such as multiplication, transposing and Gaussian elimination for calculating null spaces are involved. However, Step 3 can be time consuming, since random elements are chosen until we find an element with a nontrivial null space of low dimension, preferably of dimension one. The advantage of finding an element with a null space of dimension one is that it is sufficient to run the third step of the algorithm only for one vector.

However, finding an element with a nonzero null space is not always easy. The probability calculations have been done in [7]. In the present work we use a method described in [7] to generate elements which have nonzero nullspace.

2.2 The Generalized Norton Criterion

This method is a generalization of the Norton Criterion and is introduced in [7]. The authors' strategy is as follows:

Algorithm 2.2.1. ***The Generalized Norton Criterion***

Input: *As before we start with a matrix representation* δ *of* A *over* $\mathbb{F}$ *in terms of matrices* $\{\delta(a_1), \ldots, \delta(a_k)\}$ *for a generating set* $\{a_1, \ldots, a_k\}$ *of* A*,*

1. *Pick an element a in A.*

2. *Calculate the characteristic polynomial $c(x)$ of $\delta(a)$.*

3. *Factor $c(x) = p_1(x)^{a_1} p_2(x)^{a_2} \cdots p_n(x)^{a_n}$ into irreducible factors. For each irreducible factor $p_i(x)$ for $1 \leq i \leq n$, do the following:*

   i) Set $b = p_i(\delta(a))$.

   ii) Calculate the null space of b, $N(b)$. If $\dim_{\mathbb{F}}(N(b)) = \deg(p_i(x))$, then we call $p_i(x)$ a ***good factor*** *of $c(x)$.*

   iii) Choose a nonzero vector v in $N(b)$ and calculate a basis of the A-submodule vA by using the spinning algorithm. If this is a proper submodule return the matrices with respect to this basis of vA for the generating system on vA.

   iv) If $vA = V$, calculate the null space N of b^T.

   v) Choose a nonzero vector v^* in $N(b^T)$ and calculate a basis of the left A-submodule Av^*. If this is a proper submodule then return the transposed matrices with respect to this basis for the generating system on Av^*.

   vi) If $p_i(x)$ is a good factor, then return the answer 'irreducible'.

4. *Go back to Step 1.*

Output*: Either the information that V is irreducible, or a matrix representation of A on an A-submodule W.*

Remark: Note that the only additional calculations compared with the Norton Criterion are calculating and factoring the characteristic polynomial.

In the generalized Norton algorithm we claim that if $p_i(x)$ is a good factor then it is sufficient to run the algorithm only for one nonzero vector $v \in N(b)$. In other cases, examination of a single vector will not give a conclusive test for irreducibility, but might prove reducibility. If not, another factor of $c(x)$ or another element $\delta(a)$ is selected. Let us explain why this generalized algorithm is much more efficient.

Claim 2.2.2. *Let A be a finite dimensional algebra over a finite field $\mathbb{F}$, δ a matrix representation of A over $\mathbb{F}$, $a \in A$ and $c(x)$ be the characteristic polynomial of $\delta(a)$. Let $c(x) = p_1(x)^{a_1} p_2(x)^{a_2} \cdots p_n(x)^{a_n}$ be a factorization of $c(x)$ into irreducible polynomials. Then $\dim(N(p_i(\delta(a))) = \deg(p_i(x))$ for some $1 \leq i \leq n$ if and only if the multiplicity of p_i in the minimal polynomial of $\delta(a)$ is a_i.*

Proof. Suppose for some fixed i, $1 \leq i \leq n$, the multiplicity of $p_i(x)$ in the minimal polynomial of $\delta(a)$ is a_i. Then after a base change $\delta(a)$ is equal to the following matrix

$$\begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & A_n \end{pmatrix},$$

where each A_k is a block diagonal matrix with each block being the companian matrix of p_k^t for $1 \leq t \leq a_k$. Since the multiplicity of $p_i(x)$ in the minimal polynomial of $\delta(a)$ is the same as the multiplicity in the characteristic polynomial of $\delta(a)$, A_i consists of one block, i.e. A_i is the companion matrix of the polynomial $p_i^{a_i}$ and

$$\dim N(p_j(A_i)) = \deg(p_i) \cdot \delta_{ij}.$$

Then $p_i(\delta(a))$ is similar to

$$p_i(\delta(a)) = \begin{pmatrix} p_i(A_1) & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & p_i(A_2) & \cdots & \cdots & \cdots & 0 \\ \vdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \cdots & \cdots & p_i(A_i) & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & p_i(A_n) \end{pmatrix}$$

and since $\dim N(p_j(A_i)) = \deg(p_i) \cdot \delta_{ij}$,

$$\dim N(p_i(\delta(a))) = \deg(p_i(x)).$$

The converse direction is trivial. □

Remark 2.2.3. *A special case occurs if $a_i = 1$, i.e. the multiplicity of $p_i(x)$ is one. Then*

$$p_i(\delta(a)) = \begin{pmatrix} [0]_{k\times k} & \cdots & \cdots & [0]_{k\times k} \\ 0 & A_2(x) & \cdots & 0 \\ \vdots & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & A_n(x) \end{pmatrix},$$

where $k = \deg(p_i(x))$.

Definition 2.2.4. *Let A be a finite dimensional algebra over a finite field $\mathbb{F}$, δ a matrix representation of A over $\mathbb{F}$, $a \in A$ and $c(x)$ the characteristic polynomial of $\delta(a)$. Let $c(x) = p_1(x)^{a_1} p_2(x)^{a_2} \cdots p_n(x)^{a_n}$ be a factorization of $c(x)$ into irreducible polynomials. A factor $p_i(x)$ of $c(x)$ is called a* ***very good factor****, if its multiplicity in the characteristic polynomial is one, i.e. if $a_i = 1$. Note that every very good factor is a good factor.*

As we promised, we want to compare the Norton Criterion with the Generalized Norton Criterion. As usual, let V be an A-module with the corresponding representation

$$\delta : A \longrightarrow \mathrm{End}_{\mathbb{F}}(V)$$

Recall that in the Norton Criterion, we are interested in $N(\delta(a))$ for $a \in A$, whereas in the Generalized Norton Criterion, we look at $N(p(\delta(a)))$, where $p(x)$ is an irreducible factor of the characteristic polynomial of $\delta(a)$. It follows that the Norton Criterion is just a special case of the Generalized Norton Criterion. Suppose the characteristic polynomial of $\delta(a)$ has the irreducible factor $p(x) = x$. Then $N(p(\delta(a))) = N(\delta(a))$ and if $p(x) = x$ is a good factor, then $\dim(N(\delta(a))) = 1$ by Claim 2.2.2 and it is enough to run the algorithm for one single vector $v \in N(\delta(a))$ up to scalars.

2.3 Examples

In this section we give two examples to see how the algorithm 2.2.1 works.

Example 1: Let $G = S_3$ and $\mathbb{F} = \mathbb{F}_2$ be the field with two elements. Our goal is to determine all the irreducible representations of S_3 over $\mathbb{F}_2$. We start with the natural permutation representation δ of S_3 defined as

$$\delta : S_3 \longrightarrow \mathrm{GL}(3, \mathbb{F})$$

where

$$(1 \quad 3 \quad 2) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = A_1,$$

and

$$(1 \quad 2) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = A_2,$$

where $\{(1 \quad 3 \quad 2), (1 \quad 2)\}$ is a generating set for the group algebra $\mathbb{F}_2 S_3$.

Step 1: Pick an element $\delta(a)$, which corresponds to a random element a in the group algebra $\mathbb{F}_2 S_3$ using our generating set A. Let

$$\delta(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Step 2: The characteristic polynomial of $\delta(a)$ is $x^3 + 1 = (x + 1)(x^2 + x + 1)$, where each factor is irreducible.

Step 3: Let $p(x) = x^2 + x + 1$. Then $b = p(\delta(a)) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

Step 4: The null space of b is $N(b) = \{(k, l, m) | k + l + m = 0;\ k, l, m \in \mathbb{F}_2\}$.

Step 5: Let $v_1 = (1 \quad 1 \quad 0) \in N(b)$. We have to find a basis B of $v_1 A$. Applying A_1 to v_1 we get $v_2 = v_1 A_1 = (1 \quad 0 \quad 1)$. Since v_1 and v_2 are linearly independent, we include v_2 to B. Since $v_2 A_2 = v_2$, we can go to the next step, namely we have to apply A_1 and A_2 to v_2.

$$(1 \quad 0 \quad 1) A_1 = (0 \quad 1 \quad 1),$$

$$(1 \quad 0 \quad 1) A_2 = (0 \quad 1 \quad 1),$$

Note that our basis set is $B = \{(1 \quad 1 \quad 0), (1 \quad 0 \quad 1)\}$, since $(0 \quad 1 \quad 1)$ is just the sum of those two vectors.

Let $W = \text{Span}_{\mathbb{F}}\{(1 \quad 1 \quad 0), (1 \quad 0 \quad 1)\}$.Then W is a $\mathbb{F}S_3$-submodule of dimension 2.

Let us check whether W is simple. Since we have a basis for W, we can write explicitly the corresponding degree two representation. Let $v_1 = (1 \quad 1 \quad 0)$ and $v_2 = (1 \quad 0 \quad 1)$. Then

$$v_1 A_1 = v_2, \quad v_2 A_1 = v_1 + v_2,$$

$$v_1 A_2 = v_1, \quad v_2 A_2 = v_1 + v_2.$$

Therefore the representation with respect to $\{v_1, v_2\}$ is

$$\delta' \colon S_3 \to GL(2, \mathbb{F})$$

where

$$(1 \quad 3 \quad 2) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = A_1',$$

$$(1 \quad 2) \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = A_2'.$$

We will follow the same steps as above for our new representation to see whether it is irreducible. First we need to pick an element $\delta'(a)$. Let

$$\delta'(a) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Then the characteristic polynomial of $\delta'(a)$ is $(x+1)(x+1)$. Let $p(x) = x + 1$.

$$b = p(\delta'(a)) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Hence the null space of b, $N(b) = \{(k \quad 0) | k \in \mathbb{F}_2\}$. Let $v = (0 \quad 1) \in N(b)$. Then

$$vA_1 = (1 \quad 1).$$

So $B_1 = \{(1 \quad 0), (0 \quad 1)\}$ is a basis for $vA = W$. By the algorithm, we have to follow the same steps for the dual space V^*.

The null space $N(b^T) = \{(0 \quad k)|k \in \mathbb{F}_2\}$ and using the spinning algorithm with $(1 \quad 0)^T$ we conclude that $B_2 = \{(1 \quad 0)^T, (0 \quad 1)^T\}$ is a basis for $A^*v = V^*$. Therefore this degree 2 representation of S_3 over $\mathbb{F}_2$ is irreducible.

Since we started with a degree 3 representation, i.e. a module of dimension 3, and found a degree 2 irreducible representation we get a quotient of dimension 1.

Recall that $W = \text{Span}_{\mathbb{F}}\{(1 \quad 1 \quad 0), (1 \quad 0 \quad 1)\}$. Then a complement of W, $\hat{W} = \{(k \quad k \quad k)|k \in \mathbb{F}_2\}$ is the 1 dimensional irreducible submodule, corresponding to the trivial representation.

To summarize, we found 2 irreducible representations of S_3 over $\mathbb{F}_2$. The trivial representation and a degree 2 irreducible representation.

Example 2: We consider $G = S_3$ and $\mathbb{F} = \mathbb{F}_3$. We will see how changing the underlying field effects the whole problem even in this small example. Let

$$\delta : S_3 \to GL(3, \mathbb{F})$$

be the natural permutation representation of S_3 where

$$(1 \quad 3 \quad 2) \mapsto A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

and

$$(1 \quad 2) \mapsto A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $\delta(a) = A_1$. Then the characteristic polynomial of $\delta(a)$ is $-(x+2)^3$. Let $p(x) = x + 2$ and

$$b = p(\delta(a)) = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

The null space of b is, $N(b) = \{(k \quad k \quad k)|k \in \mathbb{F}_3\}$. Let $v \in N(b) = (1 \quad 1 \quad 1)$. Then vA is a one dimensional irreducible subspace.

Now let $\delta(a) = A_2$. The characteristic polynomial of $\delta(a)$ is $-(x-1)^2(1+x)$. If we choose $p(x) = (x+1)$ then

$$p(\delta(a)) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Hence $N(p(\delta(a))) = \{k \quad 2k \quad 0) : k \in \mathbb{F}_3\}$. Following the algorithm, we see that for $v = (1 \quad 2 \quad 0) \in N(p(\delta(a)))$, vA is a one dimensional subspace. Since $\mathbb{F}_3$ is a splitting field for S_3 and S_3 has two conjugacy classes of 3-regular elements, these are the only irreducible representations of S_3 up to equivalence over $\mathbb{F}_3$ by Theorem 1.5.7.

Chapter 3

MODULE CATEGORIES AND CONDENSATION

In the last chapter we have seen direct methods to analyze a given matrix representation of a finite dimensional algebra A over a finite field $\mathbb{F}$. The purpose of this chapter is the description of the methods based on the relationship between the module category of A and the module category of a subalgebra of A. Such techniques have been applied algorithmically by Richard Parker and Thackray first and they have been documented by Thackray in his Ph.D. thesis, see [19]. We first start with the basic ideas of category theory that will be needed in discussing the equivalence of the category of finitely generated $\mathbb{F}G$-modules and the category of finitely generated modules for a subalgebra of $\mathbb{F}G$, where G is a finite group. An in-depth discussion of this material can be found in [1].

3.1 Morita Equivalence

Definition 3.1.1. *A* ***category*** $\mathcal{C}$ *is given by:*

1. *a class of objects* $\mathrm{Obj}(\mathcal{C})$*.*

2. *for* A, B *in* $\mathrm{Obj}(\mathcal{C})$*, a set* $\mathcal{C}(A, B)$ *called the morphisms from* A *to* B *such that if* $A \neq C$ *and* $B \neq D$ *then* $\mathcal{C}(A, B) \cap \mathcal{C}(C, D)$ *is empty.*

3. *for* $A, B, C \in \mathrm{Obj}(\mathcal{C})$ *there is a mapping*

$$\mathcal{C}(A, B) \times \mathcal{C}(B, C) \to \mathcal{C}(A, C), \;\; (f, g) \mapsto gf$$

which is called composition and which satisfies:

A1. The associative law: $h(gf) = (hg)f$ *for all* $f \in \mathcal{C}(A,B)$, $g \in \mathcal{C}(B,C)$, $h \in \mathcal{C}(C,D)$.

A2. The existence of identities: To each object A *there is a morphism* 1_A *from* A *to* A *such that, for all* $f \in \mathcal{C}(A,B)$, $g \in \mathcal{C}(C,A)$,

$$f1_A = f, \ \ 1_A g = g.$$

Within a category $\mathcal{C}$ we have the morphism sets $\mathcal{C}(A,B)$ which establish connections between the objects A and B. We now formulate the notion of a transformation from one category to another.

Definition 3.1.2. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be categories. A* ***covariant functor*** $F\colon \mathcal{C} \to \mathcal{D}$ *is a rule which assigns to each object* A *of* $\mathcal{C}$ *an object* $F(A) \in \mathrm{Obj}(\mathcal{D})$ *and to each morphism* $f \in \mathcal{C}(A,B)$ *a morphism* $F(f) \in \mathcal{D}(F(A), F(B))$, *such that*

$$F(fg) = F(f)F(g),$$

and

$$F(1_A) = 1_{F(A)}.$$

Here we give an example.

Example 3.1.3. *Let* A *and* B *be finite dimensional algebras over a field* $\mathbb{F}$. *We define the category of finitely generated right* A*-modules,* $\mathrm{RMod}(A)$ *as follows: The objects are the finitely generated right* A*-modules and the morphisms are the* A*-module homomorphisms,* $\mathrm{Hom}_A(M,N)$, *for* $M, N \in \mathrm{RMod}(A)$. *Throughout this dissertation we assume that each functor* F *from* $\mathrm{RMod}(A)$ *to* $\mathrm{RMod}(B)$ *is additive, i.e. for* $M, N \in \mathrm{RMod}(A)$, *and morphisms* $f, g\colon M \to N$,

$$F(f+g) = F(f) + F(g).$$

We also assume that all functors from $\mathrm{RMod}(A)$ *to* $\mathrm{RMod}(B)$ *are* $\mathbb{F}$*-linear, namely*

$$F(\alpha f + \beta g) = \alpha F(f) + \beta F(g),$$

for all $\alpha, \beta \in \mathbb{F}$.

Definition 3.1.4. *Let* F, G *be two functors from the category* $\mathcal{C}$ *to the category* $\mathcal{D}$. *Then a* ***natural transformation*** t *from* F *to* G *is a rule assigning to each object* $M \in \mathrm{Obj}(\mathcal{C})$ *a morphism* $t_M \colon F(M) \to G(M)$ *in* $\mathcal{D}$ *such that for any morphism* $f \colon M \to N$ *in* $\mathcal{C}$, *the following diagram commutes.*

$$\begin{array}{ccc} F(M) & \xrightarrow{t_M} & G(M) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(N) & \xrightarrow{t_N} & G(N) \end{array}$$

If t_M *is an isomorphism for each* M *then* t *is called* ***natural equivalence*** *and we write* $F \cong G$.

Definition 3.1.5. *Let* A *and* B *be finite dimensional algebras over a field* $\mathbb{F}$. *Let* $\mathrm{RMod}(A)$ *and* $\mathrm{RMod}(B)$ *be the categories of finitely generated right* A *and* B*-modules, respectively. Then a covariant functor*

$$F : \mathrm{RMod}(A) \to \mathrm{RMod}(B)$$

is a ***categorical equivalence*** *if there is a covariant functor*

$$G : \mathrm{RMod}(B) \to \mathrm{RMod}(A)$$

and natural equivalences

$$GF \cong 1_{\mathrm{RMod}(A)} \quad \text{and} \quad FG \cong 1_{\mathrm{RMod}(B)}.$$

$\mathrm{RMod}(A)$ *and* $\mathrm{RMod}(B)$ *are called* ***equivalent*** *in case there exists two functors with the above properties. The algebras* A *and* B *are called* ***Morita equivalent*** *algebras and we say that* F *and* G *form a* ***Morita equivalence***.

The following lemma summarizes some consequences of Morita equivalence.

Lemma 3.1.6. *Let A and B be Morita equivalent algebras with*

$$F\colon \operatorname{RMod}(A) \to \operatorname{RMod}(B)$$

and

$$F\colon \operatorname{RMod}(B) \to \operatorname{RMod}(A)$$

forming this equivalence. Then the following hold for $M \in \operatorname{RMod}(A)$.

1. *M is projective if and only if $F(M)$ is projective.*

2. *M is simple (semisimple) if and only if $F(M)$ is simple (semisimple).*

3. *M is indecomposable if and only if $F(M)$ is indecomposable.*

Furthermore the lattice of submodules of M is isomorphic to the lattice of of submodules of $F(M)$. This implies that $F(\operatorname{Soc}(M)) = \operatorname{Soc}(F(M))$,and hence F maps socle series of M to the socle series of $F(M)$.

Proof. A proof of this lemma can be found in [1] on pages 254-258. □

Corollary 3.1.7. *Let A and B be Morita equivalent algebras. Then the number of isomorphism classes of simple modules is the same for A and B.*

Proof. A proof of this lemma can be found in [1] on pages 254-258. □

3.2 Sufficient Conditions for Morita Equivalence

In the last section, we saw some of the consequences of two finite dimensional algebras A and B over a field $\mathbb{F}$ being Morita equivalent. In this section, we outline sufficient conditions for A and B to be Morita equivalent. The types of functors we can naturally use for defining a Morita equivalence are the following:

Definition 3.2.1. *Let A and B be finite dimensional algebras over a field $\mathbb{F}$, let M be an A-B-bimodule and let N be a B-A-bimodule.*

1. *The **tensor functor*** $- \otimes_A M\colon \operatorname{RMod}(A) \to \operatorname{RMod}(B)$ *is defined by*

$$V \mapsto V \otimes_A M$$

for $V \in \operatorname{RMod}(A)$ *and*

$$\alpha \mapsto (V \otimes_A M \to W \otimes_A M\colon v \otimes m \mapsto \alpha(v) \otimes m)$$

for $W \in \operatorname{RMod}(A)$, $\alpha \in \operatorname{Hom}_A(V, W)$ *and* $m \in M$.

2. *The* Hom ***functor*** $\operatorname{Hom}_A(N, -)\colon \operatorname{RMod}(A) \to \operatorname{RMod}(B)$ *is defined by*

$$V \mapsto \operatorname{Hom}_A(N, V)$$

and

$$\alpha \mapsto (\operatorname{Hom}_A(N, V) \to \operatorname{Hom}_A(N, W)\colon \alpha \mapsto \alpha\lambda)$$

for $V, W \in \operatorname{RMod}(A)$ *and* $\alpha \in \operatorname{Hom}_A(V, W)$, $\lambda \in \operatorname{Hom}_A(N, V)$.

We now state a criterion that ensures that a certain pair of tensor functors realizes Morita equivalence.

Theorem 3.2.2. *Let A and B be finite dimensional algebras over $\mathbb{F}$ with P an A-B-bimodule and Q an B-A-bimodule. If $P \otimes_B Q \cong A$ as A-A-bimodules and $Q \otimes_A P \cong B$ as B-B-bimodules, then the functors $(- \otimes_A P)$ and $(- \otimes_B Q)$ define a Morita equivalence between A and B.*

Proof. For a proof see [12, page 43]. □

Definition 3.2.3. *For a module category* $\operatorname{RMod}(A)$, *an element* $P \in \operatorname{RMod}(A)$ *is called a **progenerator** if it satisfies the following:*

- *P is a projective A-module,*
- *every A-module is a homomorphic image of a direct sum of copies of P.*

Corollary 3.2.4. *Let P be a progenerator for* $\mathrm{RMod}(A)$ *and* $Q = \mathrm{Hom}_A(P, A)$*. Then* $E = \mathrm{End}_A(P)$ *is Morita equivalent to A and a categorical equivalence is given by the functors* $(- \otimes_A Q)$ *and* $(- \otimes_E P)$*.*

Proof. A proof of this corollary can be found in [12, page 43]. □

The following theorem is very powerful. It states that given two equivalent algebras via inverse equivalences F and G, these functors are naturally equivalent to either Hom or tensor functor.

Theorem 3.2.5. *(Morita's Theorem). Let A and B be Morita equivalent algebras over a field $\mathbb{F}$ with the functors* $F\colon \mathrm{RMod}(A) \to \mathrm{RMod}(B)$ *and* $G\colon \mathrm{RMod}(B) \to \mathrm{RMod}(A)$ *forming a Morita equivalence between A and B. Then $P = F(A)$ is an A-B-bimodule and $Q = G(B)$ is an B-A-bimodule such that $P \otimes_B Q \cong A$ as A-A-bimodules and $Q \otimes_A P \cong B$ as B-B-bimodules. Furthermore, the following statements hold:*

1. *P is a progenerator for* $\mathrm{RMod}(B)$*.*
2. *Q is a progenerator for* $\mathrm{RMod}(A)$*.*
3. *There are bimodule isomorphisms*

$$Q \cong \mathrm{Hom}_A(P, A) \cong \mathrm{Hom}_B(P, B),$$

$$P \cong \mathrm{Hom}_A(Q, A) \cong \mathrm{Hom}_B(Q, B).$$

4. *There are $\mathbb{F}$-algebra isomorphisms*

$$A \cong \mathrm{End}_B(P) \cong \mathrm{End}_B(Q),$$

$$B \cong \mathrm{End}_A(P) \cong \mathrm{End}_A(Q).$$

5. *The functors $- \otimes_A P$ and $- \otimes_B Q$ define a Morita equivalence between A and B.*

6. *The functors* $\mathrm{Hom}_A(Q, -)$ *and* $\mathrm{Hom}_B(P, -)$ *define a Morita equivalence between* A *and* B.

Proof. For a proof see [1, page 262].

□

3.3 Condensation and Idempotents

By Corollary 3.2.4, the algebras A and $\mathrm{End}_A(P)$ are Morita equivalent if P is a progenerator for $\mathrm{RMod}(A)$. However determining explicitly a nontrivial progenerator for $\mathrm{RMod}(A)$ is not easy. We will restrict our attention to the situation where P is isomorphic to eA for a suitable idempotent $e \in A$ and hence $\mathrm{End}_A(P)$ is isomorphic to the subalgebra eAe. This approach makes our life simpler in practice, since finding nontrivial idempotents in A is easier than determining a progenerator for $\mathrm{RMod}(A)$.

Definition 3.3.1. *Let* A *be an* $\mathbb{F}$*-algebra and let* $e \in A$ *be an idempotent. The subalgebra* eAe *of* A *is called the* ***condensation subalgebra*** *of* A *corresponding to* e *and we define the restriction functor*

$$\mathrm{res}^A_{eAe}\colon\ \mathrm{RMod}(A) \to \mathrm{RMod}(eAe)$$

by

$$V \mapsto Ve$$

for $V \in \mathrm{RMod}(A)$ *and for* $\alpha \in \mathrm{Hom}_A(V, W)$

$$\alpha \mapsto \alpha|_{Ve} \in \mathrm{Hom}_{eAe}(Ve, We).$$

Remark 3.3.2. *If* A *is an* $\mathbb{F}$*-algebra and* e *is an idempotent. then* $eA \otimes_A Ae \cong eAe$ *as* eAe*-*eAe*-bimodules hence the composition of the two tensor functors* $(- \otimes_A Ae)$ *and* $(- \otimes_{eAe} eA)$ *is naturally equivalent to* $1_{\mathrm{RMod}(eAe)}$*. Moreover, the functor* $(- \otimes_A Ae)$ *and the restriction functor* res^A_{eAe} *are naturally equivalent and we can replace the functor* $(- \otimes_A Ae)$ *by* res^A_{eAe}.

In Theorem 3.2.2 and Corollary 3.2.4 we saw a relationship between progenerators and tensor functors forming Morita equivalence. The following theorem realizes this relationship in terms of idempotents and characterizes under which conditions the two tensor functors $(- \otimes_{eAe} eA)$ and $(- \otimes_A Ae)$ form a Morita equivalence for A and eAe.

Theorem 3.3.3. *Let $e \in A$ be an idempotent, $e_1A, \dots, e_kA$ be representatives of the isomorphism classes of projective indecomposable A-modules, and $S_1, \dots, S_k$ be representatives of the isomorphism classes of simple A-modules such that $e_iA/\operatorname{Rad}(e_iA) \cong S_i$, see Theorem 2.3.3. Then the following statements are equivalent.*

1. *The functors $(- \otimes_{eAe} eA)$ and $(- \otimes_A Ae)$ form a Morita equivalence for A and eAe.*

2. *AeA is equal to A.*

3. *For all $1 \leq i \leq k$, the A-module e_iA is a direct summand of eA.*

4. *$S_ie \neq 0$ for all $1 \leq i \leq k$.*

Proof. A proof of this theorem can be found in [12, page 46]. □

Definition 3.3.4. *Let $e \in A$ be an idempotent. If e satisfies one of the conditions in Theorem 3.3.3, then e is called a for A* ***faithful idempotent****.*

3.4 Condensation of group algebras

In this section we focus on the situation where A is a group algebra. Let G be a finite group and $\mathbb{F}$ a finite field. Our main interest is in finding a faithful idempotent in $\mathbb{F}G$ so that $\mathbb{F}G$ and $e\mathbb{F}Ge$ are Morita equivalent. In group algebras certain idempotents are easy to construct as described in the following lemma.

Lemma 3.4.1. *Let H be a subgroup of G such that the characteristic of $\mathbb{F}$ does not divide the order of H. Then*

$$e_H := \frac{1}{|H|}\sum_{h\in H} h$$

is an idempotent in $\mathbb{F}G$.

Proof. Clearly $e_H \in \mathbb{F}G$, so we only have to show that e_H is an idempotent, i.e. $e_H^2 = e_H$.

$$\begin{aligned} e_H^2 &= \frac{1}{|H|}\sum_{h_1\in H} h_1 \frac{1}{|H|}\sum_{h_2\in H} h_2 \\ &= \frac{1}{|H|^2}\sum_{h_1\in H} h_1 \sum_{h_2\in H} h_2 \\ &= \frac{1}{|H|^2}\sum_{h_1\in H}\sum_{h_2\in H} h_1 h_2 \\ &= \frac{1}{|H|}\sum_{h\in H} h = e_H \end{aligned}$$

which proves the lemma. □

Lemma 3.4.2. *Let $\mathbb{F}$ be a field with characteristic p, p a prime, H be a subgroup of G such that $p \nmid |H|$ and $e_H \in \mathbb{F}G$ the idempotent defined in Lemma 3.4.1. Then for all $V \in \mathrm{RMod}(\mathbb{F}G)$*

$$Ve_H = V^H = \{v \in V | vh = v \text{ for all } h \in H\}.$$

Proof. Clearly $Ve_H \subseteq V^H$ since $e_H \cdot h = e_H$ for any $h \in H$. Conversely, let $v \in V^H$, then

$$ve_H = \frac{1}{|H|}\sum_{h\in H} mh = \frac{1}{|H|}\sum_{h\in H} m = \frac{|H|}{|H|}m = m,$$

so $v \in Ve_H$. □

We are interested in a subgroup H of G, such that the idempotent $e_H = \frac{1}{|H|}\sum_{h\in H} h$ is a faithful idempotent in $\mathbb{F}G$.

Definition 3.4.3. *We call a subgroup H of G a faithful condensation subgroup for $\mathbb{F}G$ if $e_H\mathbb{F}Ge_H$ and $\mathbb{F}G$ are Morita equivalent.*

Given a group G and a field $\mathbb{F}$ with characteristic p our first task for showing Morita equivalence is to determine the faithful condensation subgroups of G. The following theorem gives a character theoretic condition, which ensures that $e_H\mathbb{F}Ge_H$ is Morita equivalent to $\mathbb{F}G$.

Theorem 3.4.4. *Let G be a group, $\mathbb{F}$ a splitting field of G with charactersitic p, H a subgroup of G where $p \nmid |H|$, $e_H \in \mathbb{F}G$ is the idempotent defined in Lemma* 3.4.1*. The tensor functors $(- \otimes_{e_H\mathbb{F}Ge_H} e_H\mathbb{F}G)$ and $(- \otimes_{\mathbb{F}G} \mathbb{F}Ge_H)$ form a Morita equivalence between* $\mathrm{RMod}(\mathbb{F}G)$ *and* $\mathrm{RMod}(e_H\mathbb{F}Ge_H)$ *if and only if for all $\phi \in IBr(G)$, the character theoretic scalar product of the restriction of ϕ to H with the trivial character 1_H is nonzero, i.e.*

$$(1_H, \phi|_H)_H \neq 0.$$

Proof. For a proof see [12, page 53]. □

Next we will give a slightly different formulation of this criterion, that we use in practice for finding faithful condensation subgroups.

Lemma 3.4.5. *Let G be a group, $\mathbb{F}$ be a splitting field of G with charactersitic p. Let $S_1, \cdots S_n$ be representatives for the simple $\mathbb{F}G$-modules. Furthermore, let $\Phi_1, \cdots \Phi_n$ be the projective indecomposable characters corresponding to the projective covers of $S_1, \cdots S_n$,(see Definition* 2.5.8*) and let H be a subgroup of G such that p does not divide $|H|$. Then e_H is a faithful idempotent if and only if the permutation character $1_H^G = \sum_{i=1}^{n} c_i\Phi_i$ with each $c_i > 0$ an integer for all $1 \leq i \leq n$. Moreover, the simple $e_H\mathbb{F}Ge_H$-modules are given by S_ie_H and the dimension of the simple $e_H\mathbb{F}Ge_H$-module S_ie_H is given by the multiplicity of Φ_i in 1_H^G for $1 \leq i \leq n$.*

Proof. For a proof see [6, page 63].

□

Let V be an $\mathbb{F}G$-module. The advantage of condensation follows from the fact that the $e_H\mathbb{F}Ge_H$-module Ve_H consists of the fixed points of the action of H on V. Hence the dimension of Ve_H is usually much smaller than the dimension of V. We still have to answer the following questions about condensing the $\mathbb{F}G$-module V with a faithful idempotent e_H.

- How do we find a basis of the $e_H\mathbb{F}Ge_H$-module Ve_H?
- How do we find the action of the generators of $e_H\mathbb{F}Ge_H$ on Ve_H?

We answer the above questions for a permutation $\mathbb{F}G$-module V.

Proposition 3.4.6. *Let H be a subgroup of G, let $e_H = \frac{1}{|H|}\sum_{h\in H} h$ and let Ω be a G-set with H-orbits $O_1, \ldots, O_r$. Let V the corresponding $\mathbb{F}G$-permutation module with permutation basis b_k, $k \in \Omega$. Then*

$$\bar{B} = \{\bar{O}_1, \ldots, \bar{O}_r\}$$

is a basis for Ve_H, the fixed space of H on V, where

$$\bar{O}_i = \sum_{k\in O_i} b_k.$$

Moreover, the action of e_Hge_H for $g \in G$ on Ve_H with respect to this basis is given by

$$\bar{O}_i e_H g e_H = \sum_{j=1}^{r} c_{ij} \frac{1}{|O_j|} \bar{O}_j,$$

where

$$c_{ij} = |\{k \in O_i | kg \in O_j\}|.$$

Furthermore, for a fixed i,

$$|\{c_{ij} | c_{ij} \neq 0, 1 \leq j \leq r\}| \leq |H|.$$

Proof. It is obvious that $\bar{O}_i$'s are fixed by all elements of $h \in H$. Let v be an element in Ve_H, then $v = \sum_{k\in\Omega} \alpha_k b_k$ with $\alpha_k \in \mathbb{F}$. Since $vh = v$, we have that $\alpha_{k_1} = \alpha_{k_2}$ when $k_1, k_2 \in O_i$, thus $v = \sum_{i=1}^{r} \alpha_i \bar{O}_i$. So the set $\bar{B}$ spans Ve_H. Linear independence follows from the fact that the H-orbits $O_1, \ldots, O_r$ are a partition of the G-set Ω. Note further that if b_k is an element of the permutation basis of V such that $k \in O_j$, then the product

$$b_k e_H = \frac{1}{|H|} \sum_{h\in H} b_k h = \frac{1}{|H|} |H_k| \sum_{k\in O_j} b_k, \tag{3.1}$$

where $H_k = \{h \in H | b_k h = b_k\}$. Therefore

$$b_k e_H = \frac{1}{|O_j|} \sum_{k\in O_j} b_k = \frac{\bar{O}_j}{|O_j|} \tag{3.2}$$

Hence using (4.2),

$$\begin{aligned} \bar{O}_i e_H g e_H &= \bar{O}_i g e_H = \left(\sum_{k\in O_i} b_k \right) g e_H = \left(\sum_{k\in O_i} b_k g \right) e_H = \left(\sum_{k\in O_i} b_{kg} \right) e_H \\ &= \sum_{k\in O_i, kg\in O_j} \frac{\bar{O}_j}{|O_j|} = \sum_{j=1}^{r} c_{ij} \frac{\bar{O}_j}{|O_j|}, \end{aligned}$$

where

$$c_{ij} = |\{o_i \in O_i | o_i g \in O_j\}|.$$

To finish the proof, we notice that if $c_{ij} \neq 0$, then there is a k in O_i such that $kg \in O_j$. But there are at most $|H|$ elements in O_i, so the size of the set $\{c_{ij} | c_{ij} \neq 0, 1 \leq j \leq r\}$ is bounded by the order of H.

□

The following example explicitly shows how to calculate the c_{ij}'s of Proposition 3.4.6 and how to construct a matrix for the action of $e_H g e_H$ on Ve_H for $g \in G$.

Example 3.4.7. *Let $G = S_4 = \langle(12), (1234)\rangle$, $\mathbb{F}_3$ the field with 3 elements and H the subgroup generated by the permutations $(12), (34)$. Take Ω be the set $\{1, 2, 3, 4\}$. The subgroup H has 2 orbits in Ω, $O_1 = \{1, 2\}$ and $O_2 = \{3, 4\}$. The coefficients c_{ij}'s of Proposition 3.4.6 are*

$$c_{11} = 1, c_{12} = 1, c_{21} = 1, c_{22} = 1$$

for the permutation (1234). Weighting c_{ij} for all $1 \leq i, j \leq 2$ by $|O_i|^{-1} \mod 3$ we get the matrix

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

for the action of $e_H(1234)e_H$ on the fixed space of H of this $\mathbb{F}_3S_4$-permutation module.

In the rest of this section we study the condensation algebra $e_H\mathbb{F}Ge_H$ for $e_H = \frac{1}{|H|}\sum_{h\in H} h$ in further detail.

The following theorem gives a method to calculate the dimension of $e_H\mathbb{F}Ge_H$ using character theory.

Proposition 3.4.8. *Let H be a faithful condensation subgroup of G. Then*

$$\dim_{\mathbb{F}}(e_H\mathbb{F}Ge_H) = (1_H^G, 1_H^G).$$

Proof. For a proof see [6, page 70]. □

The next theorem, whose proof was given by M. Wiegelmann in his diploma thesis [20], gives an important criterion which ensures that a set of elements in $e_H\mathbb{F}Ge_H$ generates the algebra $e_H\mathbb{F}Ge_H$.

Theorem 3.4.9. *Let $N \leq G$ be a subgroup which normalizes H and suppose that N/H is a p-group generated by $\{n_1H, \ldots, n_lH\}$, $l \in \mathbf{N}$. Let $f = \sum_{n\in N} n \in \mathbb{F}G$ and denote by $V = f\mathbb{F}G$ the permutation module of G on the cosets of N. Given elements*

$g_1, \ldots, g_k, k \in \mathbf{N}$, consider the subalgebra B of $e_H\mathbb{F}Ge_H$ generated by $e_Hg_1e_H, \ldots, e_Hg_ke_H$. Then the following statements holds. Suppose that $f \cdot B = Ve_H$ then

$$\{e_Hn_1e_H, \ldots, e_Hn_le_H, e_Hg_1e_H, \ldots, e_Hg_ke_H\}$$

is a generating set for the condensation algebra $e_H\mathbb{F}Ge_H$.

Proof. For a proof see [12, page 53]. □

3.5 Peakword Condensation

Throughout this section A is a finite dimensional algebra over a finite field $\mathbb{F}$ and $e \in A$ is an idempotent. In this section we will exploit the functorial relationship between RMod(A) and RMod(eAe) in a new direction. Instead of aiming at a Morita equivalence we will choose e to be a primitive idempotent in A. This approach is used for determining submodule lattices [14], for determining the socle series and the radical series of a given A-module [15] and also for computing the homomorphisms between A-modules [16].

All of the approaches are based on special words in A called peakwords, which were first introduced in [14]. These words help us to find primitive idempotents in A.

In this section we give a short overview of the notation, define peakwords and describe main results from [14]. The theory that we will summarize can be applied to any finite dimensional algebra A. The reader should keep in mind that we are mainly interested in applying the techniques to an $e_H\mathbb{F}Ge_H$-module Ve_H, where V is a projective $\mathbb{F}G$-permutation module, $\mathbb{F}$ is a finite field, H is a faithful condensation subgroup of a finite group G and e_H is a faithful idempotent in $\mathbb{F}G$.

Let S be a simple A-module and let V be an A-module. For $a \in A$ we denote by $\mathrm{Ker}_V(a)$ the kernel of the $\mathbb{F}$-endomorphism induced on V by a. Similarly $\mathrm{Im}_V(a)$ denotes the image of the $\mathbb{F}$-endomorphism induced on V by a.

Definition 3.5.1. *An A-module V is called S-**local** if $V/\operatorname{Rad}(V) \cong S$ and an idempotent $e \in A$ is called S-**primitive** if the module eA is S-local.*

Example 3.5.2. *Recall that by Theorem 1.2.8 together with Definition 1.2.9, the projective cover $P(S)$ of a simple A-module S, has the property that*

$$P(S)/\operatorname{Rad}(P(S)) \cong S.$$

Hence $P(S)$ is S-local.

Definition 3.5.3. *An element a_S in the $\mathbb{F}$-algebra A is called an S-**peakword** with respect to V for a compostion factor S of V, if the following conditions are fulfilled.*

- $\ker_T(a_s) = \{0\}$ *for all composition factors T of V not isomorphic to S.*
- $\dim_{\mathbb{F}}(\ker_S(a_S^2)) = [\operatorname{End}_A(S) : \mathbb{F}]$.

Definition 3.5.4. *Let S be a simple A-module and let $M(V)$ be the lattice of submodules of the A-module V. We denote by $M_S(V)$ the subset of A-submodules of W, such that $W/\operatorname{Rad}(W)$ is isomorphic to a direct sum of copies of S. An element of $M_S(V)$ is called an S-radical module.*

The following theorem shows that $M_S(V)$ is controlled locally.

Theorem 3.5.5. *Let $e \in A$ be an S-primitive idempotent. Then the mapping*

$$\kappa\colon M_S(V) \to M(Ve)\colon W \mapsto We$$

is an isomorphism of lattices. Its inverse is given by

$$\kappa^{-1}\colon M(Ve) \to M_S(V)\colon \tilde{W} \mapsto \tilde{W}A.$$

Proof. For a proof see [14]. □

The following theorem describes how to find the S-primitive idempotents in A using S-peakwords.

Theorem 3.5.6. *Let A be an finite dimensional $\mathbb{F}$-algebra, V be a faithful A-module and let $a \in A$. Then there exists a uniquely determined idempotent $e \in A$ with the following property: If $V = \mathrm{Ker}_V(a^k) \oplus \mathrm{Im}_V(a^k)$ is the Fitting decomposition of V with respect to the endomorphism induced by a, (see* 1.2.4*) , then $\mathrm{Ker}_V(a^k) = Ve$ and $\mathrm{Im}_V(a^k) = V(1-e)$. Here $k \geq 0$ is an integer such that for all $n \geq k$, $\mathrm{Ker}_V(a^n) = \mathrm{Ker}_V(a^k)$. If a is an S-peakword then e is an S-primitive idempotent with respect to V and $\mathrm{Ker}_V(a^k)$ is called the stable kernel of the S-peakword a.*

Proof. For a proof see [14].

□

Given an A-module V, the following lemma shows a way to find the projective indecomposable summands of V.

Lemma 3.5.7. *Let V be an A-module and S a composition factor of V. Let π be an A-epimorphism from V onto the S-local A-module M and let e_S be an S-primitive idempotent. Then the image of $\pi(v)$ of $v \in Ve_S$ doesn't generate M if and only if $\pi(v) \in \mathrm{Rad}_A(M) \cap Me_S = \mathrm{Rad}_{e_s A e_s}(Me_s)$. Hence the subspace of vectors v in Ve_S, for which $\pi(v)$ doesn't generate M, has codimension $\dim_{\mathbb{F}}(\mathrm{End}_A(S)) = \dim_{\mathbb{F}}(Se_S)$. Furthermore, if $M \cong P(S)$, the projective cover of S, then the ratio of vectors in Ve_S that do not generate an A-submodule isomorphic to $P(S)$ to the vectors in Ve_S is at most $\frac{1}{|\mathrm{End}_A(S)|}$.*

Proof. For a proof see [14]. □

The theory we described in this section is called peakword condensation. Peakword condensation can be applied to any A-module V. However as we mentioned earlier we will use peakword condensation to construct the projective direct summands of a projective $e_H\mathbb{F}Ge_H$-module Ve_H, where V is a projective permutation $\mathbb{F}G$-module. Hence we apply peakword condensation to the $e_H\mathbb{F}Ge_H$-module Ve_H and we will describe this in the next section.

3.6 The projective indecomposable modules of a group and implementing construction of the P.I.M.s of a group

In this section we will describe how the theory we introduced in the previous sections can be used to compute the projective indecomposable modules of a finite group over a finite field. The algorithms that are used in this thesis are implemented in GAP and the C-MeatAxe, see [18]. GAP (Groups, Algorithms, Programming) is a system for computational discrete algebra, with particular emphasis on computational group theory. The C-MeatAxe is a set of programs for working with matrices over finite fields. Its primary purpose is the calculation of representations, although it can be used for other purposes, such as investigating subgroup structure of a group and the structure of a given module.

Throughout this section G denotes a finite group and $\mathbb{F}$ denotes a finite field. The following algorithm uses implementations in the C-MeatAxe of the fixed point condensation and peakword condensation that are introduced in [12] and [6].

Algorithm 3.6.1. *(Computing the P.I.M.s of a finite group G.)*

Input: *A group G and a prime number p which divides the order of G.*

Computation:

1. *Find a faithful condensation subgroup H of G, such that the algebras $\mathbb{F}G$ and $e_H\mathbb{F}Ge_H$ are Morita equivalent.*

2. *Find a generating set for the condensation subalgebra $e_H\mathbb{F}Ge_H$.*

3. *Construct a projective permutation $\mathbb{F}G$-module V.*

4. *Condense V using the subgroup H we have chosen in Step 1.*

5. *Compute peakwords corresponding to each simple $e_H\mathbb{F}Ge_H$-module Ve_H to construct the projective indecomposable summands of Ve_H.*

Output: *The projective indecomposable summands of Ve_H.*

We should note that after running this algorithm once, unless we constructed all the P.I.M.s of G, we go back to Step 3 and construct another projective permutation $\mathbb{F}G$-module W that has some projective indecomposable summands that we have not constructed yet. We repeat this until we construct all the P.I.M.s of G.

In the rest of this section we will describe each step in detail and mention the programs already written using GAP and the C-MeatAxe. All the programs in GAP can only be applied to groups that are in the GAP data library.

In GAP, we use a special generating system of the group G that is in the GAP library, namely a so-called system of standard generators as defined by R.A. Wilson in [21]. For the groups in the GAP library, permutation representations and matrix representations are constructed in the standard generators of these groups and are accessible in the GAP library via the GAP package `atlasrep`. This library also includes words in the standard generators, namely expressions of group elements as an iterated product of standard generators, that give generators of all subgroups up to conjugacy of groups that are in the GAP data library.

Let G be a group that is in the GAP data library.

Step1: Find a condensation subgroup H of G:

As we proved in Lemma 3.4.5, the group H is a condensation subgroup of G if and only if the permutation character 1_H^G contains each of the projective indecomposable characters at least once. One can use GAP to find possible condensation subgroups of G. To understand how this is done we use the table of marks of G from the GAP library.

Definition 3.6.2. *Let $L(G) = \{H_1, \ldots, H_n\}$ be a complete set of representatives of the set of conjugacy classes of subgroups of a finite group G.*
(a) If M is any G-set then the function

$$m_M \colon L(G) \to \mathbf{Z}, \qquad H \mapsto |Fix_M(H)|$$

is called the ***mark*** *of M, where $|Fix_M(H)|$ denotes the number of elements in M that are fixed by H.*
(b) The ***table of marks*** *of G is the square matrix*

$$M(G) = [m_{G/H_i}(H_j)].$$

We call H_i the representative of the ith conjugacy class of the table of marks of G. For a subgroup H that is conjugate to H_i we say that H is in the ith conjugacy class of the table of marks of G.

The GAP command `TableOfMarks` takes as an input a group G that is in the GAP data library and returns its table of marks. The implementation of the table of marks in GAP makes it possible to construct each representing subgroup H_i of G with number i in the table of marks of G using the standard generators of G. The generators of H_i are given as words in the standard generators and this information can be accessed via the straight line programs. Straight line programs in GAP are programs that describe an efficient way of evaluating a word in a group. The GAP command `StraightLineProgsramsTom` takes as an input the table of marks of a group and returns a list that contains at position i a list of straight line programs encoding the generators of H_i, the respresentative of the ith conjugacy class of the table of marks of G.

The character table of a group G is one of the necessary ingredients we need for the implementations of the algorithms we use. The GAP command `CharacterTable` takes as an input a group G and returns the ordinary character table of G. If

`CharacterTable` is called with a group G and a prime number p, then it returns the p-modular character table of G.

Using the table of marks and the character table of G we can get all the transitive permutation characters on the cosets of the subgroups H_i of G using `PermCharsTom` in GAP.

The GAP command `PermCharsTom` takes two inputs, the character table and the table of marks of a group G. It returns the list of transitive permutation characters on the cosets of the representatives H_i of the table of marks of G.

The command `Decomposition` enables us to find the decomposition of the permutation characters $1_{H_i}^G$ restricted to the p-regular classes of G into the projective indecomposable characters of G restricted to the p-regular classes of G. If $1_{H_i}^G$ contains each of the projective indecomposable characters of G and $p \nmid |H_i|$ then H_i is a condensation subgroup by Lemma 3.4.5. In this case any subgroup that is conjugate to the representative H_i of the ith conjugacy class of the table of marks of G is also a condensation subgroup. In the first step we only decide on the number i of the representative we choose for condensation. In the next step we construct the condensation subgroup H that is conjugate to H_i.

Step2: Finding a generating set for the condensation subalgebra: The second step is to find a generating set for the condensation subalgebra $e_H\mathbb{F}Ge_H$. We first determine a representative subgroup that normalizes a subgroup in the ith conjugacy class of the table of marks of G where i is the number we have chosen in the first step. The GAP command `NormalizerTom` takes the table of marks and the number i of the i-th conjugacy class of the table of marks of a group as input and finds the conjugacy class with number j of the normalizer N in G of a subgroup U in the ith conjugacy class of the table of marks of G. We construct the representative H_j of the jth conjugacy class using the straight line programs. Without loss of generality let us call this subgroup N. Finally we construct a subgroup H in G whose generators will be words in the generators of N so that H is normal in N and H is in the ith conjugacy class of the table of marks of G. The subgroup H is the condensation subgroup we are going to use. If N/H a p-group, then we can apply Theorem 3.4.9: If $n_1, \dots n_r$ are generators of N and $e_H g_1 e_H, \dots, e_H g_s e_H$ are elements in $e_H \mathbb{F}G e_H$ such that the first standard basis vector $e_1 \in 1_N^G e_H$ spans the whole space $1_N^G e_H$ under the action of $e_H g_1 e_H, \dots, e_H g_s e_H$ then $e_H g_1 e_H, \dots, e_H g_s e_H, e_H n_1 e_H, \dots, e_H n_r e_H$ generate the condensation subalgebra $e_H \mathbb{F}G e_H$. This requires constructing the permutation $\mathbb{F}G$-module 1_N^G and condensing it to get the $e_H \mathbb{F}G e_H$-module $1_N^G e_H$.

In the C-MeatAxe, a permutation module can be condensed with the program `kd.sh`. The implementation of this program uses the ideas described in Theorems 3.4.6 and Theorem 3.4.6.

Step3: Construct projective permutation FG-modules and condense them:

Let V be a projective permutation $\mathbb{F}G$-module. We condense V with the condensation subgroup H as defined in Step 2 using the C-MeatAxe command `kd.sh`. Then we run the C-MeatAxe program `chop` to determine the composition factors of the $e_H \mathbb{F}G e_H$-module Ve_H which are in one to one correspondence with the composition factors of the $\mathbb{F}G$-module V by Morita equivalence.

Step4: Apply peakword condensation to $e_H \mathbb{F}G e_H$-module Ve_H:
The C-MeatAxe program `pwkond` applied to the $e_H \mathbb{F}G e_H$-module Ve_H computes a peakword a_S for each composition factor S of the module Ve_H and it also finds a basis for the stable kernel in Ve_H of the peakword a_S (see Theorem 3.5.6). Suppose the projective cover $P(S)$ of S is a direct summand of Ve_H. By Lemma 3.5.7, a random vector v in the stable kernel in Ve_H of the peakword a_S generates an $e_H \mathbb{F}G e_H$-submodule isomorphic to $P(S)$ with high probability. So once we run `pwkond` the only task we have is to choose a random vector from the stable kernel of a_S and use the C-MeatAxe routine `zsp` to construct the smallest $e_H \mathbb{F}G e_H$-submodule containing this vector until we get a submodule of dimension equal to the dimension of $P(S)$.

The C-MeatAxe program `zsp` is the implementation of the spinning algorithm we discussed in Section 3.1.

Once we have all the projective indecomposable $e_H \mathbb{F} G e_H$-modules of G, their socle series can be computed with the C-MeatAxe program `soc`. The details about this program can be found in [15] or in [12].

Chapter 4

A NEW TECHNIQUE FOR COMPUTING THE P.I.M.S OF LARGE FINITE GROUPS AND IMPLEMENTATION

In this chapter we want to describe a different technique for computing the projective indecomposable modules of a given group G. In [15] the authors used both fixed point condensation and peakword condensation to compute the projective indecomposable modules of the Mathieu group M_{23}, using the theory and computational tools we described in the previous chapter. However for larger groups, these techniques are out of reach since the matrices corresponding to the actions of the generators of the condensation subalgebra do not fit into memory.

Throughout this chapter G denotes a finite group, $\mathbb{F}$ a finite field, H a condensation subgroup of G and e_H is the corresponding faithful idempotent. As we proved in Lemma 3.4.6, the matrix corresponding to the action of $e_H g e_H \in e_H \mathbb{F} G e_H$ on a $e_H \mathbb{F} G e_H$-module has at most $|H|$ many nonzero entries per row. In [6], the author called such a matrix row sparse and implemented a row sparse format in the C-MeatAxe to take advantage of these special matrices and solved the memory issue.

The algorithm we will describe in this chapter for constructing the P.I.M.s of a group G is implemented by the author in GAP and uses the author's own row sparse matrices. First we discuss this row sparse matrix in GAP format.

4.1 Sparse Matrices in GAP

The simplest form of a matrix over a finite field $\mathbb{F}$ of order less than 256 is given in GAP as a list of lists. In this form each entry of the matrix takes 1 byte in space. An example of a GAP matrix in this form is the following:

Let

$$m = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

be a matrix with entries from the field $\mathbb{F}_2$ with 2 elements. In GAP, m is defined as follows:

```
gap> m:=[[0,0,1,1],[1,1,0,0],[0,1,1,0],[1,0,1,0]]*Z(2)^0;;
```

Definition 4.1.1. *Suppose m is an $n \times n$ row sparse matrix over a field $\mathbb{F}$ with at most h many nonzero entries per row. A **sparse GAP matrix** is a list of n lists, where the ith list corresponds to the ith row of m, and consists of 2 lists of length h, for $1 \leq i \leq n$. The first contains the nonzero entries from $\mathbb{F}$ of the row i and the second consists of the positions of these entries.*

The matrix m above as a sparse GAP matrix with 2 nonzero entries per row would be defined as:

```
gap> m:=[[[1,1]*Z(2)^0,[3,4]],[[1,1]*Z(2)^0,[1,2]],[[1,1]*Z(2)^0,[2,3]],
[[1,1]*Z(2)^0,[1,3]]];;
```

What is the advantage of storing them in this fashion? Let us calculate how much space a sparse GAP matrix takes. Suppose we are given an $n \times n$ sparse matrix m with at most h many nonzero entries per row, which is a list of n lists by the above definition. The ith list consists of 2 lists and the length of both lists are h. For the first of these lists we need h bytes. The second of these lists consists of the positions of the nonzero entries. Keeping the large sizes of the matrices we use in our computations in mind, the numbers in the second list are large integers and we assume we need 4 bytes to store a large integer in GAP. Hence for the second list we need $4 * h$ bytes. In total the ith list takes $5h$ bytes in space so to store the matrix m

in GAP we need $5 * h * n$ bytes. If we stored m as a GAP matrix in its simple form then we would need n^2 bytes. In the cases we consider, M_{24} and A_{12}, see Chapter 5, h is considerably smaller than n. Hence working with matrices that take $5 * h * n$ bytes certainly has advantages than working with matrices that need n^2 bytes.

The following example describes the situation:

Example 4.1.2. *Let $G = M_{24}$ and $\mathbb{F}_2$ be the field with 2 elements. H is a condensation subgroup of M_{24} with order* 27*. K_1 is a subgroup of G, with $|K_1| = 55$. The permutation $e_H \mathbb{F} G e_H$-module $1_{K_1}^G e_H$ has dimension* 164864*, see chapter 5. The matrices corresponding to the actions of the generators of the condensation subalgebra $e_H \mathbb{F} G e_H$ on $1_{K_1}^G e_H$ have* 164864 *rows and have at most 27 nonzero entries per row. Storing such a matrix in GAP as a simple GAP matrix format would require approximately* $164864^2 \cong 27$ *gigabytes. On the other hand one needs* $5 * 27 * 164864 \cong 22$ *megabytes to store the same matrix in the author's sparse GAP matrix format.*

As we will see in Chapter 5, we will be using words that are sums of products of the condensation subalgebra generators in our computations. Since the matrices for the actions of the elements in the condensation algebra on $e_H \mathbb{F} G e_H$-modules are row sparse we prefer to define such words given in sparse GAP matrices. We describe the situation in an example. Suppose a, b are the generators of the condensation subalgebra and $w = a * a + a * b$ is a word in these generators. We define the word w in GAP as a list of lists, i.e. `w:=[[a,a],[a,b]]`.

Definition 4.1.3. *A word w that is defined in GAP as described above is called a* ***sparse GAP word****.*

The sparse GAP matrices and the sparse GAP words as we defined above are essential in our computations for constructing the P.I.M.s of a group. Another basic notion is the order polynomial of a vector.

Lemma 4.1.4. *Let V be a finite dimensional vector space over an arbitrary field $\mathbb{F}$, and let T be a fixed linear transformation in $\mathrm{End}_{\mathbb{F}}(V)$ and $v \in V$ nonzero. Then there exists a monic nonzero polynomial in $\mathbb{F}[x]$*

$$p(x) = x^d - c_{d-1}x^{d-1} - \cdots - c_0,$$

$c_i \in \mathbb{F}$*, such that*

$$vp(T) = 0.$$

Proof. Since V is finite dimensional, there exists an integer $d \geq 1$ such that $\{v, vT, \ldots, vT^{d-1}\}$ is linearly independent, and such that $\{v, vT, \ldots, vT^{d-1}, vT^d\}$ are linearly dependent. Then there exist elements $c_0, \ldots, c_{d-1}$ in $\mathbb{F}$ such that

$$vT^d = c_0 v + c_1 vT + \cdots + c_{d-1} vT^{d-1}.$$

Letting

$$p(x) = x^d - c_{d-1}x^{d-1} - \cdots - c_0,$$

we have

$$vp(T) = 0.$$

□

Definition 4.1.5. *The monic polynomial $p(x)$ with the smallest degree satisfying the conditions of the above lemma is called the* ***order polynomial*** *of v.*

Let V be a permutation $\mathbb{F}G$-module with composition factors $S_1 \ldots, S_n$. Let H be a faithful condensation subgroup of G and e_H a faithful idempotent so that $\mathbb{F}G$ and $e_H\mathbb{F}Ge_H$ are Morita equivalent. Then each composition factor S_i of V condenses to a composition factor S_ie_H of the $e_H\mathbb{F}Ge_H$-module Ve_H. The C-MeatAxe program `pwkond` applied to the $e_H\mathbb{F}Ge_H$-module Ve_H finds an S_ie_H-peakword $a_{S_ie_H} \in \mathrm{End}(Ve_H)$, and computes a basis for the stable kernel of $a_{S_ie_H}$ for all $1 \leq i \leq n$. If the projective cover $P(S_ie_H)$ of S_ie_H is a direct summand of Ve_H, then as we stated in

Lemma 3.5.7, a vector in the stable kernel of $a_{S_i e_H}$ in Ve_H spins up to the projective cover of $S_i e_H$ with high probability. The order polynomial $p_i(x)$ of a vector $v \in Ve_H$ for $a_{S_i e_H}$ enables us to compute a vector in the stable kernel in Ve_H of $a_{S_i e_H}$ for all $1 \leq i \leq n$.

Without loss of generality let S be one of the composition factors $S_i e_H$ of the $e_H \mathbb{F} G e_H$-module Ve_H and a_S be an S peakword. Suppose $p(x)$ is the order polynomial of a vector $v \in Ve_H$ for an S-peakword $a_S \in \operatorname{End} Ve_H$. We factorize $p(x)$ as $p(x) = x^r q(x)$ with $r \geq 0$ such that $q(x)$ is not divisible by x. Plugging in the peakword $a_S \in \operatorname{End} Ve_H$ in the polynomial $p(x)$, we get $p(a_S) = a_S^r q(a_S)$. Hence we have

$$vp(a_S) = vq(a_S)a_S^r = 0.$$

Here is a modified algorithm that computes the projective indecomposable modules of a group G from the GAP data library over a finite field $\mathbb{F}$. We assume we already have chosen a condensation subgroup and we have a generating set for the condensation subalgebra $e_H \mathbb{F} G e_H$.

Algorithm 4.1.6. ***Computing the P.I.M.s of a group:***

Input: *A projective permutation* $\mathbb{F}G$*-module* $1_K^G = \oplus_{i=1}^m c_i P(S_i)$ *with* $c_i > 0$ *integer for* $1 \leq i \leq m$ *where* $K \leq G$ *and peakwords for the simple* $\mathbb{F}G$*-modules* S_i*.*

1. *Compute the actions of the generators of the condensation subalgebra* $e_H \mathbb{F} G e_H$ *on* $1_K^G e_H$ *as sparse GAP matrices, see Definition* 4.1.1*.*
2. *In GAP generate a random vector* $v \neq 0$ *in* $1_K^G e_H$*.*
3. *Define the peakword* $a_{S_i e_H} \in \operatorname{End} Ve_H$ *for each* $S_i e_H$ *as a sparse GAP word, see Definition* 4.1.3*.*
4. *Calculate the order polynomial* $p(x)$ *of* v *for* $a_{S_i e_H}$ *described above.*

5. *Compute the factor $q(x)$ that is not divisible by x of $p(x)$ and compute $vq(a)$.*

6. *Compute the $e_H FGe_H$-submodule generated by $vq(a)$ of dimension equal to the dimension of the projective cover of $S_i e_H$.*

Output: *The projective indecomposable $e_H\mathbb{F}Ge_H$-modules $P(S_i e_H)$.*

4.2 Program Descriptions

In this section we describe the programs written by the author to implement the Algorithm 4.1.6 described at the end of the last section. The source code can be found in the Appendix and also at the author's website: `math.arizona.edu/~kalaycioglu`. To run these programs we need the following setup: G is a finite group from the GAP data library and H is a faithful condensation subgroup of G. We need the H-suborbits of G described in Proposition 3.4.6, which can be found by the GAP operation `FindSuborbits` from the `orb` package in GAP. We have a generating set for the condensation subalgebra $e_H\mathbb{F}Ge_H$ and we also have a projective permutation module 1_K^G on the cosets of a subgroup K of G. For each composition factor S of 1_K^G we know the S-peakword a_S.

MyOrbitIntersectionMatrix: This program is a variation of the `OrbitIntersectionMatrix` program in the GAP package `orb`. As input it takes the H-suborbits of G and an element $g \in G$. It outputs a sparse GAP matrix for the action of $e_H g e_H$ on the $e_H\mathbb{F}Ge_H$-module $1_K^G e_H$.

MyVectorTimesSparseMat: This program multiplies a vector $v \in 1_K^G e_H$ by a sparse GAP matrix and returns a vector.

VectorTimesWord: This program multiplies a vector $v \in 1_K^G e_H$ by a sparse GAP word and returns a vector.

For example let a, b be the sparse GAP matrices . Let w be the word $w = a{*}b{+}a{*}a$. Then the sparse GAP word w is defined as `w=[[a,b],[a,a]]`. This program computes:

```
MyVectorTimesSparseMat(MyVectorTimesSparseMat(v,a),b)+
MyVectorTimesSparseMat(MyVectorTimesSparseMat(v,a),a).
```

MySpinningwwords: Let a_{Se_H} be a peakword for the composition factor Se_H of $1_K^G e_H$ and let w be the sparse GAP word corresponding to a_{Se_H}. This program takes as an input a vector $v \in 1_K^G e_H$ and the sparse GAP word w and returns a list consisting of the coefficients of the order polynomial $p(x)$ of v for $a_{Se_H} \in \mathrm{End}(1_K^G e_H)$.

MyVecttimesPolyinMatRingwwords: Let a_{Se_H} be a peakword for the composition factor Se_H of $1_K^G e_H$, $p(x)$ the order polynomial of a vector $v \in 1_K^G e_H$ for $a_{Se_H} \in \mathrm{End}(1_K^G e_H)$, $q(x) \neq 1$ the factor of $p(x)$ that is not divisible by x and let w be the sparse GAP word corresponding to a_{Se_H}. This program takes as an input the vector v, the coefficient list of $q(x)$ and the sparse GAP word w. It returns a vector in the stable kernel of a_{Se_H}.

MySpinningwgen: Let a_{Se_H} be a peakword for the composition factor Se_H of $1_K^G e_H$. This program takes as an input a vector v in the stable kernel of a_{Se_H} in $1_K^G e_H$ returned from `MyVecttimesPolyinMatRingswwords`, a list of sparse GAP matrices for the actions of the generators of the condensation subalgebra on $1_K^G e_H$, and an integer l that is the dimension of the projective cover of Se_H. The program computes the smallest $e_H \mathbb{F} G e_H$-submodule of $1_K^G e_H$ containing v. If the dimension of this submodule is not equal to l then the program returns `fail`, otherwise it outputs a matrix corresponding to the action of each generator on $1_K^G e_H$.

Chapter 5
RESULTS

In this chapter we will give explicit examples for Morita equivalences between the group algebra $\mathbb{F}G$ and a subalgebra $e_H\mathbb{F}Ge_H$, where H is a faithful condensation subgroup of G. We will use the equivalence to determine the layers of the socle series of the projective indecomposable $\mathbb{F}G$-modules. The groups we examine in this chapter are the Mathieu group M_{24} and the alternating group A_{12}. Both of these groups are in the GAP data library. Throughout this chapter we use a special generating system for these groups namely a so-called system of standard generators as defined by R.A. Wilson [21]. For the Mathieu group M_{24} and the alternating group A_{12}, permutation representations and matrix representations are constructed in the standard generators of these groups and are accessible in the GAP library via the GAP package `atlasrep`. This library also includes words in the standard generators that give generators of all subgroups up to conjugacy of these groups. These words will be used in the examples we describe in this chapter. Finally, given two elements a, b in a group G we use standard words called $z_1 = z1(a,b), \ldots, z_{10} = z10(a,b)$ to form new elements in G derived from a and b, whose definition is given as follows: $z_1 = z_1(a,b) = a$, $z_2 = z_2(a,b) = b$, $z_3 = z_1 * z_2$, $z_{10} = z_3 * z_2$, $z_4 = z_3 * z_{10}$, $z_5 = z_3 * z_4$, $z_6 = z_3 * z_5$, $z_7 = z_4 * z_5$, $z_8 = z_3 * z_6$, $z_9 = z_7 * z_{10}$.

5.1 The P.I.M.s of M_{24} in characteristic 2

In this section we will use fixed point condensation 3.4, peakword condensation 3.5 and the author's Algorithm 4.1.6 for computing the projective indecomposable modules of the sporadic Mathieu group M_{24} over the field $\mathbb{F}_2$ with two elements. We will also give the socle series of the P.I.M.s of M_{24}. For the rest of this section $\mathbb{F} = \mathbb{F}_2$ and

$G = M_{24}$. We will use the standard generators `g1,g2` for M_{24}, where

$$\texttt{g1} := (1,4)(2,7)(3,17)(5,13)(6,9)(8,15)(10,19)(11,18)(12,21)$$
$$(14,16)(20,24)(22,23,)$$
$$\texttt{g2} := (1,4,6)(2,21,14)(3,9,15)(5,18,10)(13,17,16)(19,24,23).$$

We begin by reviewing the main facts about the simple $\mathbb{F}G$-modules.

Lemma 5.1.1. *There are* 13 *simple* $\mathbb{F}G$*-modules, which we denote by* $1a, 11a, 11b, 44a, 44b, 120a, 220a, 220b, 252a, 320a, 320b, 1242a, 1792a$*, where the number in front of each letter corresponds to the dimension of the simple* $\mathbb{F}G$*-module. The splitting field for* M_{24} *in characteristic* 2 *is* $\mathbb{F}_2$*.*

Proof. For a proof see [10]. □

This can be checked using the GAP library of Brauer characters. The character table of M_{24} can be called from the GAP library by:

```
gap> m24:=CharacterTable("M24");;
```

The 2-modular character table of M_{24} is accessible in GAP via:

```
gap> m24mod2:=CharacterTable(m24,2);;
```

The degrees of the irreducible Brauer characters are:

```
gap> List(Irr(m24mod2),x->x[1]);;
```

We continue by identifying the subgroup H of M_{24} we are going to use for condensation. We do this in the following way. We give the number of the conjugacy class of subgroups in the GAP library of the table of marks in which the subgroup H is contained.

Lemma 5.1.2. *Let* H *be a subgroup of* M_{24} *of order* 27 *that is in the* 303*th conjugacy class of the table of marks of* M_{24}*. Then* e_H *is a faithful idempotent of* $\mathbb{F}G$*. The condensation subalgebra* $e_H FG e_H$ *has dimension* 336224*. Moreover the following table gives the dimension of the condensed simple* $\mathbb{F}G$*-modules.*

TABLE 5.1. Dimensions of the simple $\mathbb{F}_2M_{24}$-modules and the simple $e_H\mathbb{F}_2M_{24}e_H$-modules

Dimensions of the simple $\mathbb{F}G$-modules												
1	11	11	44	44	120	220	220	252	320	320	1242	1792
Dimensions of the condensed simple $e_H\mathbb{F}Ge_H$-modules												
1	1	1	2	2	6	6	6	8	8	8	46	64

Proof. Let 1_H^G be the permutation $\mathbb{F}G$-module on the cosets of H. We use the same notation for the permutation character of G that is afforded by 1_H^G. By Lemma 3.4.5, in order to prove that e_H is a faithful idempotent we have to show that 1_H^G contains each of the projective indecomposable characters of G. Also by Lemma 3.4.5, the dimension of a simple e_HFGe_H-module is given by the multiplicity of the projective indecomposable character of G in 1_H^G which can be computed as described before. Finally by Proposition 3.4.8, the dimension of $e_H\mathbb{F}Ge_H$ is given by the scalar product $(1_H^G, 1_H^G)$. In GAP this can be done as follows:

We first access the table of marks of M_{24} in GAP.

```
gap> tom:=TableOfMarks("M24");;
```

The 2-decomposition matrix of M_{24} as defined in 1.5.6 can be computed via:

```
gap> d:= DecompositionMatrix(m24mod2);;
```

The following line in GAP determines the irreducible characters of M_{24}.

```
gap> irr:=Irr(m24);;
```

The projective indecomposable characters of M_{24} can be obtained as follows in GAP:

```
gap> ipr:=TransposedMat(d)*irr;;
```

We get all the transitive permutation characters:

```
gap> permchar:=PermCharsTom(m24,tom);;
```

The permutation characters corresponding to the permutation modules on the cosets of subgroup representatives of M_{24} of odd order can be decomposed in terms of the projective indecomposable characters of M_{24}, since these permutation modules are projective by 1.2.14.

```
gap> cond:=Decomposition(ipr,permchar,5);;
```

Note that since H is conjugate to the subgroup with number 303 from the table of marks, the decomposition of the permutation character 1_H^G in terms of the projective indecomposable characters of M_{24} can be obtained via

```
gap> cond[303];
gap> [1,1,1,2,2,6,6,6,8,8,8,46,64]
```

□

We proceed by listing the dimensions of the projective indecomposable modules of M_{24} and the dimensions of the projective indecomposable $e_H\mathbb{F}Ge_H$-modules.

Lemma 5.1.3. *There are* 13 *projective indecomposable modules of* M_{24} *with the following dimensions:*

TABLE 5.2. Dimensions of the projective indecomposable $\mathbb{F}_2M_{24}$-modules and the dimensions of the projective indecomposable $e_H\mathbb{F}_2M_{24}e_H$-modules

P.I.M.s	P(1a)	P(11a)	P(11b)	P(44a)	P(44b)	P(120a)	P(220a)
Dim.	269312	293888	293888	193536	193536	168960	102400
Cond. Dim.	10080	10992	10992	7264	7264	6320	3808

P.I.M.s	P(220b)	P(252a)	P(320a)	P(320b)	P(1242a)	P(1792a)
Dim.	102400	125952	46080	46080	4506	21504
Cond. Dim.	3808	4704	1696	1696	1664	784

Proof. By Theorem 1.2.8 there is a one to one correspondence between the simple $\mathbb{F}G$-modules and the projective indecomposable $\mathbb{F}G$-modules. The dimensions of the P.I.M.s of M_{24} and their condensed dimensions can be calculated as follows:

```
gap> dimproj:=CartanMat("M24")*cond[1];;
gap> dimcondproj:=CartanMat("M24")*cond[303];;
```

□

The next step is finding a generating set for the condensation subalgebra $e_H\mathbb{F}Ge_H$.

Lemma 5.1.4. *Let N be the representative of the 1162th conjugacy class of the table of marks with generators a, b. The faithful condensation subgroup H that is in the 303th conjugacy class of the table of marks of M_{24} is generated by*

$$\{aba^2ba, a^2ba^2b, a^3baba^3bab\}$$

and is normal in N. The condensation subalgebra $e_H\mathbb{F}Ge_H$ is generated by the words $e_Hz_3e_H, e_Hz_7e_H, e_Hz_9e_H, e_Hae_H, e_Hbe_H$.

Proof. The generators of N can be constructed using the standard generators `g1,g2` of M_{24} and the straight line programs:

```
gap> progs:=StraightLineProgramsTom(tom);;
gap> a:=ResultOfStraightLineProgram(progs[1162][1],[g1,g2]);;
gap> b:=ResultOfStraightLineProgram(progs[1162][2],[g1,g2]);;
```

The subgroup H of M_{24} then is constructed via:

```
gap> h1:=a*b*a*a*b*a;;
gap> h2:=a*a*b*a*a*b;;
gap> h3:=a*a*a*b*a*b*a*a*a*b*a*b;;
gap> h:=Group(h1,h2,h3);;
```

To show that H is in the 303th conjugacy class of the table of marks of M_{24} first we construct the representative subgroup of the 303th conjugacy class using the standard generators of M_{24}.

```
gap> rep:=RepresentativeTomByGeneratorsNC(tom,303,[g1,g2]);;
```

M_{24} is constructed by:

```
gap> groupm24:=Group(g1,g2);;
```

We ask GAP whether the subgroups `h` and `res` are conjugate in `groupm24`.

```
gap> IsConjugate(groupm24,h,res);
true
```

Using GAP, the order of the subgroup N can be found as follows:

```
gap> OrdersTom(tom)[1162];
gap> 216
```

Since the order of N/H is 8, N/H is a 2-group and the hypothesis in Theorem 3.4.9 is satisfied. We proceed by constructing the permutation $\mathbb{F}G$-module on the cosets of N in the simple $\mathbb{F}G$-module `120a` as follows:

```
gap> g1:= AtlasGenerators("M24",12).generators[1];
gap> g2:= AtlasGenerators("M24",12).generators[2];
```

Here `g1,g2` are 120×120 matrices. We restrict this representation to the generators of the subgroup N, namely we find the generators of N in the irreducible representation of degree 120 using the straight line programs in GAP. We then intersect the eigenspaces of the generators of N for the eigenvalue 1. This results in a 2 dimensional subspace of vectors that are fixed under the subgroup N. The C-MeatAxe program `zvp` shows that the M_{24}-orbit of the second vector of the basis gives us the requested permutation representation of degree $1133440 = [M_{24} : N]$.

We proceed by condensing the permutation $\mathbb{F}G$-module 1_N^G using the C-MeatAxe routine `kd.sh`. This program gives the action of the elements $e_H z3 e_H$, $e_H z7 e_H$ and $e_H z9 e_H$ on the condensed permutation module $1_N^G e_H$ and it turns out that the first standard basis vector of $1_N^G e_H$ spans the whole space $1_N^G e_H$ under the action of these three condensed elements $e_H z3 e_H$, $e_H z7 e_H$ and $e_H z9 e_H$. We checked this using the C-MeatAxe routine `zsp`. So according to Theorem 3.4.9, they generate together with the elements $e_H a e_H$ and $e_H b e_H$ the full condensation algebra $e_H \mathbb{F}G e_H$.

□

We continue with constructing explicit matrix representations for the condensed projective indecomposable $\mathbb{F}G$-modules.

Lemma 5.1.5. *Let* U_1 *be a subgroup of order* 253 *with number* 1172 *from the table* *of marks of M_{24}. The permutation module of G on the cosets of U_1 is projective and splits as:*

$$1_{U_1}^G = P(1a) \oplus P(220a) \oplus P(220b) \oplus 2 * P(320a) \oplus 2 * P(320b) \oplus 4 * P(1242a) \oplus 6 * P(1792a)$$

Proof. By Theorem 1.2.14 modules induced from subgroups of odd order are projective. The decomposition of $1_{U_1}^G$ into projective indecomposable modules can be verified explicitly by using the 2-modular character table of G as follows:

Recall that `cond` is the decomposition of the permutation characters of M_{24} on the cosets of subgroups of odd order into the projective indecomposable characters of M_{24}.

```
gap> cond[1172];
gap> [1,0,0,0,0,0,1,1,0,2,2,4,6]
```
□

We first construct the permutation $\mathbb{F}G$-module on the cosets of U_1 as described in the proof of the Theorem 5.2.4. Then we condense this module with the condensation subgroup H using the C-MeatAxe routine `kd.sh`. The dimension of the $e_H\mathbb{F}Ge_H$-module $1_{U_1}^G e_H$ can be computed via character theoretic scalar product $(1_H^G, 1_{U_1}^G)$:

```
gap> ScalarProduct(permchar[303],permchar[1172]);
gap> 35840
```

The dimension of the $e_H\mathbb{F}Ge_H$-module $1_{U_1}^G e_H$ is within the range of previously written programs so the author's Algorithm 4.1.6 is not used to compute the seven projective indecomposable modules of M_{24} that are listed above. These modules and their socle series were computed before and can also be found in [6].

We analyze the $e_H\mathbb{F}Ge_H$-module $1_{U_1}^G e_H$ using the C-MeatAxe program `chop`. With this program we find the composition factors of the $e_H\mathbb{F}Ge_H$-module $1_{U_1}^G e_H$ and their multiplicities in $1_{U_1}^G e_H$. Below, we listed some of the results. The first line gives the name of the composition factors, the second line lists the multiplicity of each composition factor in the $e_H\mathbb{F}Ge_H$-module $1_{U_1}^G e_H$ and the last line gives the degree of the splitting field of the composition factors.

```
CFInfo.ConstituentNames := ["1a","1b","1c","2a","2b","6a","6b","6c",
"8a","8b","8c","46a","64a"];
CFInfo.Multiplicity := [1074,1163,1163,767,767,668,406,406,183,183,500,
178,84];
CFInfo.SplittingField := [1,1,1,1,1,1,1,1,1,1,1,1,1];
```

We next determine peakwords for the composition factors and evaluate powers of these words in the above representation until their kernel is stable using the C-MeatAxe program `pwkond`. The output of `pwkond` printed below lists for each composition factor of the $e_H\mathbb{F}G$-module $1_{U_1}^G e_H$, that we called `kdperm` in the C-MeatAxe, the following information: the number of a word w in the generators as defined by the C-MeatAxe and an irreducible factor f of the characteristic polynomial of w such that $f(w)$ is a peakword for the composition factor. Note that `a,b,c,d,e` corresponds to the generators of the condensation subalgebra that we discussed in Lemma 5.2.4.

```
*** PEAK WORD CONDENSATION ***

Peak word for kdperm8b is 70264 (edeaba+bc+cbab+bebad2+d), pol=x
Peak word for kdperm2a is 76725 (dcd2e2+eded2b+cbc2dc+becb2+bebaca), pol=x
Peak word for kdperm6b is 82664 (b3d2+badbea), pol=x+1
Peak word for kdperm8c is 89646 (ed3+dc4e+dc), pol=x
Peak word for kdperm46a is 92607 (b2aecb+c2ab3), pol=x+1
Peak word for kdperm8a is 191479 (ece2+e2daec+b), pol=x
Peak word for kdperm1b is 192819 (aedc2+bdb3e), pol=x+1
Peak word for kdperm64a is 257177 (cebdc+cbdc3+ac+bacac), pol=x+1
Peak word for kdperm1a is 348280 (cb+ed2ac2+a2cda2), pol=x+1
Peak word for kdperm1c is 438586 (ca+ca2e2+cae2c+cdeded), pol=x+1
Peak word for kdperm6a is 465545 (b+da2dcd+dad3b+ecacd), pol=x+1
```

```
Peak word for kdperm2b is 510523 (deaed2e+ceceaeb+ced), pol=x
Peak word for kdperm6c is 2907567 (bdced+d2acae2+cb2adea+bed2cae), pol=x+1
```

The next task is to find the projective indecomposable summands inside the condensed module $1_{U_1}^G e_H$. We spin up a random vector in the stable kernel of each peakword according to Lemma 3.5.7, namely the peakwords for the simples `1a,6a,6b,8b,8c,46a,64a`. This gives us the seven projective indecomposable modules for the condensed algebra, namely the projective covers of P(1a),P(6a),P(6b),P(8b), P(8c),P(46a) and P(64a). To find the socle series of these projective covers we need an analysis of the composition factors of the projective indecomposable modules and peakwords for their composition factors. Finally we apply the C-MeatAxe program `soc` to determine the socle series of the projective indecomposable modules and replace the simple $e_H\mathbb{F}Ge_H$-modules by the corresponding simple $\mathbb{F}G$-modules. The socle series of the seven projective indecomposable $\mathbb{F}G$-modules are given at the end of this section.

We still have to construct the following projective indecomposable $e_H\mathbb{F}Ge_H$-modules: P(1b), P(1c), P(2a), P(2b), P(6c), P(8a).

Lemma 5.1.6. *Let K_1 be the subgroup representative of the 603rd conjugacy class of the table of marks of M_{24}. The permutation module of G on the cosets of K_1 is projective and splits as:*

$$1_{K_1}^G = P(1a) \oplus P(11a) \oplus P(11b) \oplus 2*P(120a) \oplus 4*P(220a) \oplus 4*P(220b) \oplus 2*P(252a)$$

$$\oplus 6*P(320a) \oplus 6*P(320b) \oplus 20*P(1242a) \oplus 34*P(1792a)$$

The dimension of the $e_H\mathbb{F}Ge_H$-module $1_{K_1}^G e_H$ is 164864.

Let K_2 be the subgroup representative of the 617th conjugacy class of the table of marks of M_{24}. The permutation module of G on the cosets of K_2 is projective and *splits as:*

$$1_{K_2}^G = P(1a) \oplus P(44a) \oplus P(44b) \oplus 4 * P(120a) \oplus 3 * P(220a) \oplus 3 * P(220b) \oplus 3 * P(252a)$$

$$\oplus 2 * P(320a) \oplus 2 * P(320b) \oplus 20 * P(1242a) \oplus 28 * P(1792a)$$

The dimension of the $e_H\mathbb{F}Ge_H$*-module* $1_{K_2}^G e_H$ *is* 144160.

Proof. The orders of the subgroups K_1 and K_2 can be computed in GAP via:

```
gap> OrdersTom(tom)[606];;
gap> OrdersTom(tom)[617];;
```

By Lemma 1.2.14, since the orders of these subgroups are not divisible by the characteristic of $\mathbb{F}$, the permutation $\mathbb{F}G$-modules $1_{K_1}^G$ and $1_{K_2}^G$ are projective.

As usual we use $1_{K_1}^G$ for the permutation character afforded by the $\mathbb{F}G$-module $1_{K_1}^G$. As we described in the Proof of Lemma 5.1.2, the multiplicties of the projective indecomposable characters of M_{24} in the permutation character $1_{K_1}^G$ can be computed as:

```
gap> cond[603];
gap> [1,1,1,0,0,2,4,4,2,6,6,20,34]
```

Similarly,

```
gap> cond[617];
gap> [1,0,0,1,1,4,3,3,2,2,2,20,28]
```

gives the multiplicities of the projective indecomposable characters of M_{24} in $1_{K_2}^G$.

The dimensions of the $e_H\mathbb{F}Ge_H$-modules $1_{K_1}^G e_H$ and $1_{K_2}^G e_H$ can be found by the character theoretic scalar product

```
gap> Scalarproduct(permchar(303),permchar(606));;
gap> Scalarproduct(permchar(303),permchar(617));;
```

respectively. □

In the rest of this section we will only examine the projective permutation module $1_{K_1}^G$ for finding the socle series of the projective indecomposable summand P(11a) in detail . The other projective indecomposable summands of $1_{K_1}^G e_H$ and $1_{K_2}^G e_H$ can be found following the same steps as we will describe below so we will only give the results for the remaining projective indecomposable modules at the end of this section.

In the following we describe why the previous techniques for finding the projective indecomposable summands of the $e_H\mathbb{F}Ge_H$-module $1_{K_1}^G$ are not applicable.

The matrices corresponding to the generators of the condensation subalgebra $e_H\mathbb{F}Ge_H$ in the permutation representation of M_{24} on the cosets of K_1 are 164864×164864 matrices. Assuming that each entry of the matrix takes 1 byte in space we need 164864*164864 bytes which is approximately 27 gigabytes. By Lemma 5.2.4 the condensation subalgebra has 5 generators, hence to construct the permutation $e_H\mathbb{F}Ge_H$-module $1_{K_1}Ge_H$ we need 135 gigabytes. As we discussed in Chapter 4 these matrices would not fit into memory of the computers. By Proposition 3.4.6 the matrices corresponding to the condensation subalgebra generators have at most 27 nonzero entries in each row, where 27 is the order of the condensation subgroup H.

The first step is to construct the $e_H\mathbb{F}e_HG$-module $1_{K_1}^G e_H$. To find the actions of the generators of the condensation subalgebra $e_H\mathbb{F}Ge_H$ on the permutation module $1_{K_1}^G$ we have to use the program `MyOrbitIntersectionMatrix`.

The following lines in GAP computes the sparse GAP matrices corresponding to the actions of the generators $e_Hz3e_H, e_Hz7e_H, e_Hz9e_H, e_Hae_H, e_Hbe_H$ of the condensation subalgebra $e_H\mathbb{F}Ge_H$ on $1_{K_1}^G e_H$ respectively, where the input `sub` is the H-suborbits of G.

```
gap> a:=MyOrbitIntersectionMatrix(sub,z3)
gap> b:=MyOrbitIntersectionMatrix(sub,z7)
gap> c:=MyOrbitIntersectionMatrix(sub,z9)
gap> d:=MyOrbitIntersectionMatrix(sub,a)
```

```
gap> e:=MyOrbitIntersectionMatrix(sub,b)
```

Recall that by Morita equivalence each composition factor of the $\mathbb{F}G$-module $1_{K_1}^G$ condenses to a composition factor of the $e_H\mathbb{F}Ge_H$-module $1_{K_1}^G e_H$. The word `aedc2+bdb3e` plugged in the polynomial $f(x) = x + 1$ is a peakword for the composition factor `1b` of the $e_H\mathbb{F}Ge_H$-module $1_{K_1}^G e_H$, see page 81. This composition factor corresponds to the composition factor `11a` of the $\mathbb{F}G$-module $1_{K_1}^G$. First we define the sparse GAP word for the peakword above in terms of the sparse GAP matrices `a,b,c,d,e` corresponding to the generators of the condensation subalgebra $e_H\mathbb{F}Ge_H$:

```
gap> peakword:=[[a,e,d,c,c],[b,d,b,b,b,e],[id]];;
```

where `id` is the 164864×164864 identity matrix given in author's sparse GAP matrix format.

We generate a random vector $v \in 1_{K_1}^G e_H$ in GAP that we call `vec`. We proceed by finding the order polynomial of `vec` for `peakword` using the program `MySpinningwwords`.

```
gap> orderpoly:=MySpinningwwords(vec,peakword);;
```

The output `orderpoly` is a list consisting of the coefficients of the order polynomial $p(x)$ of `vec` for `peakword`. We are interested in the factor $q(x)$ of $p(x)$ such that $q(x)$ is not divisible by x. This can be done in GAP as:

```
gap> coefflistq:=ELMS_LIST(orderpoly,[PositionNonZero(orderpoly)..Length(orderpoly)]);;
```

The output `coefflistq` consists of the coefficients of the polynomial $q(x)$.

A vector in the stable kernel of the peakword is computed then using the following program:

```
gap>stab:= MyVecttimesPolyinMatRingswwords(vec,coefflistq,peakword);;
```

The output `stab` satisfies the equation

$$\texttt{stab} \cdot \underbrace{\texttt{peakword} \cdots \texttt{peakword}}_{k} = 0$$

According to Lemma 5.2.3, the dimension of the projective ${}_HFGe_H$-module P(1b) for the condensation algebra is 10992. Using the program `MySpinningwgen`, we spin up the vector `stab` to construct the smallest $e_H\mathbb{F}Ge_H$-invariant submodule containing `stab` given the permutation representation $1_{K_1}^G e_H$ in terms of the sparse GAP matrices `a,b,c,d,e`.

```
gap> generators:=[a,b,c,d,e];;
gap> pim1b:= MySpinningwgen(stab, gen, 10992);;
```

If the dimension of the smallest $e_H\mathbb{F}Ge_H$-invariant submodule containing `stab` is equal to 10992 then the program output `pim1b` is a list consisting of 5 matrices for the action of the condesation subalgebra generators on the condensed projective indecomposable module P(1b).

Our last task is to find the socle series of the projective indecomposable $e_H\mathbb{F}Ge_H$-module P(1b). This requires an analysis of the composition factors of P(1b) and peak-words for its composition factors. As we discussed earlier we do this analysis with the C-MeatAxe program `chop`. Hence we export the output matrices using the GAP command `CMtxBinaryFFMatOrPerm` to the C-MeatAxe. `CMtxBinaryFFMatOrPerm` takes as an input a matrix or a permutation we want to export to the C-MeatAxe, the field size and a file name for the matrix.

To export the first matrix for the condensed projective indecomposable module P(1b) to the C-MeatAxe, we do the following in GAP:

```
gap> CMtxBinaryFFMatOrPerm(pim1b[1],2,"pim1b.1");;
```

Once we have all 5 matrices in the C-MeatAxe we can use the routines `chop` and `pwkond` we described earlier to analyze the projective indecomposable $e_H\mathbb{F}Ge_H$-module P(1b). We then use the C-MeatAxe command `soc` for determining the socle series of the condensed projective indecomposable module P(1b).

Once we have the socle series of the projective indecomposable $e_H\mathbb{F}Ge_H$-module, final step requires finding the correspondence between the composition factors of the projective indecomposable $e_H \mathbb{F} G e_H$-module P(1b) and the composition factors of the projective indecomposable $\mathbb{F}G$-module P(11a). Table 5.4 gives this correspondence for all the P.I.M.s of M_{24}.

To give the reader an idea how long the programs above take for the construction of the largest projective indecomposable $e_H \mathbb{F} G e_H$-module P(1b) we give the following table:

TABLE 5.3. Computation timings

Program Name	CPU time
`MyOrbitIntersectionMatrix`	17 min
`MySpinningwwords`	4.5 hours
`MyVecttimesPolyinMatRingswwords`	8 hours
`MySpinningwgen`	3.6 days
`soc`	3 hours

Remark 5.1.7. *The computations whose timing are given in the above table are done on a computer that has a dual processor, Opteron* 246*,* 2.1 *GHz with* 8 *GB main memory running under Suse Linux* 9.1*.*

When working on this project we used the High Performance Computing systems of the University of Arizona. These are research sources, intended for testing and running large codes and parallel-processing codes. With the help of the HPC computing systems, the programs for finding the P.I.M.s of M_{24} were sent all at once , which decreased the total time we spent on this project considerably.

In the following pages we give the socle series of the P.I.M.s of M_{24}.

FIGURE 5.1. Socle series of P(1a)

1 ∗ 1a
1*11a⊕1 ∗ 44a ⊕ 1 ∗ 44b ⊕ 1 ∗ 220b ⊕ 1 ∗ 252a ⊕ 1 ∗ 1242a
4*1a⊕4 ∗ 11b ⊕ 1 ∗ 44a ⊕ 1 ∗ 44b ⊕ 1 ∗ 120a ⊕ 1 ∗ 220a ⊕ 1 ∗ 320b ⊕ 1 ∗ 320a
4*1a⊕1 ∗ 11b ⊕ 5 ∗ 11a ⊕ 2 ∗ 44a ⊕ 1 ∗ 44b ⊕ 1 ∗ 220b ⊕ 3 ∗ 120a ⊕ 1 ∗ 252a ⊕ 1 ∗ 320a ⊕ 2 ∗ 1242a ⊕ 1 ∗ 1792a
4*1a⊕3 ∗ 11b ⊕ 3 ∗ 11a ⊕ 2 ∗ 44a ⊕ 5 ∗ 44b ⊕ 1 ∗ 120a ⊕ 3 ∗ 220a ⊕ 4 ∗ 252a ⊕ 1 ∗ 320b ⊕ 2 ∗ 320a
9*1a⊕6 ∗ 11b ⊕ 3 ∗ 11a ⊕ 5 ∗ 44a ⊕ 1 ∗ 44b ⊕ 3 ∗ 220b ⊕ 4 ∗ 120a ⊕ 1 ∗ 320a ⊕ 2 ∗ 1792a
5*1a⊕6 ∗ 11b ⊕ 7 ∗ 11a ⊕ 5 ∗ 44a ⊕ 5 ∗ 44b ⊕ 2 ∗ 220b ⊕ 1 ∗ 120a ⊕ 2 ∗ 220a ⊕ 4 ∗ 252a ⊕ 1 ∗ 320b ⊕ 1 ∗ 320a
9*1a⊕10 ∗ 11b ⊕ 5 ∗ 11a ⊕ 3 ∗ 44a ⊕ 5 ∗ 44b ⊕ 1 ∗ 220b ⊕ 5 ∗ 120a ⊕ 1 ∗ 220a ⊕ 2 ∗ 320b ⊕ 2 ∗ 1242a
5*1a⊕3 ∗ 11b ⊕ 9 ∗ 11a ⊕ 4 ∗ 44a ⊕ 5 ∗ 44b ⊕ 2 ∗ 220b ⊕ 3 ∗ 120a ⊕ 3 ∗ 220a ⊕ 3 ∗ 252a ⊕ 1 ∗ 320a
9*1a⊕9 ∗ 11b ⊕ 6 ∗ 11a ⊕ 6 ∗ 44a ⊕ 5 ∗ 44b ⊕ 2 ∗ 220b ⊕ 4 ∗ 120a ⊕ 4 ∗ 220a ⊕ 2 ∗ 252a ⊕ 1 ∗ 320b ⊕ 1 ∗ 1242a
7*1a⊕6 ∗ 11b ⊕ 7 ∗ 11a ⊕ 3 ∗ 44a ⊕ 4 ∗ 44b ⊕ 4 ∗ 220b ⊕ 5 ∗ 120a ⊕ 2 ∗ 252a ⊕ 2 ∗ 320a ⊕ 2 ∗ 1242a
4*1a⊕11 ∗ 11b ⊕ 8 ∗ 11a ⊕ 6 ∗ 44a ⊕ 8 ∗ 44b ⊕ 1 ∗ 220b ⊕ 4 ∗ 120a ⊕ 5 ∗ 220a ⊕ 7 ∗ 252a ⊕ 1 ∗ 320b ⊕ 1 ∗ 320a ⊕ 2 ∗ 1242a ⊕ 1 ∗ 1792a
14*1a⊕7 ∗ 11b ⊕ 12 ∗ 11a ⊕ 4 ∗ 44a ⊕ 4 ∗ 44b ⊕ 4 ∗ 220b ⊕ 7 ∗ 120a ⊕ 1 ∗ 252a ⊕ 1 ∗ 320b ⊕ 2 ∗ 320a ⊕ 3 ∗ 1242a ⊕ 1 ∗ 1792a
4*1a⊕10 ∗ 11b ⊕ 9 ∗ 11a ⊕ 10 ∗ 44a ⊕ 10 ∗ 44b ⊕ 3 ∗ 220b ⊕ 4 ∗ 120a ⊕ 4 ∗ 220a ⊕ 6 ∗ 252a ⊕ 1 ∗ 320b ⊕ 4 ∗ 320a ⊕ 1 ∗ 1792a
16*1a⊕16 ∗ 11b ⊕ 9 ∗ 11a ⊕ 7 ∗ 44a ⊕ 7 ∗ 44b ⊕ 4 ∗ 220b ⊕ 8 ∗ 120a ⊕ 3 ∗ 220a ⊕ 1 ∗ 252a ⊕ 2 ∗ 320b ⊕ 2 ∗ 1242a ⊕ 2 ∗ 1792a
9*1a⊕8 ∗ 11b ⊕ 15 ∗ 11a ⊕ 9 ∗ 44a ⊕ 9 ∗ 44b ⊕ 1 ∗ 220b ⊕ 2 ∗ 120a ⊕ 3 ∗ 220a ⊕ 6 ∗ 252a ⊕ 2 ∗ 320b ⊕ 3 ∗ 320a ⊕ 1 ∗ 1242a
13*1a⊕13 ∗ 11b ⊕ 8 ∗ 11a ⊕ 8 ∗ 44a ⊕ 6 ∗ 44b ⊕ 3 ∗ 220b ⊕ 7 ∗ 120a ⊕ 6 ∗ 220a ⊕ 4 ∗ 252a ⊕ 2 ∗ 320b ⊕ 1 ∗ 1242a ⊕ 1 ∗ 1792a
13*1a⊕9 ∗ 11b ⊕ 12 ∗ 11a ⊕ 8 ∗ 44a ⊕ 6 ∗ 44b ⊕ 7 ∗ 220b ⊕ 7 ∗ 120a ⊕ 3 ∗ 220a ⊕ 6 ∗ 252a ⊕ 1 ∗ 320b ⊕ 2 ∗ 320a ⊕ 1 ∗ 1242a
10*1a⊕15 ∗ 11b ⊕ 8 ∗ 11a ⊕ 4 ∗ 44a ⊕ 9 ∗ 44b ⊕ 3 ∗ 220b ⊕ 9 ∗ 120a ⊕ 5 ∗ 220a ⊕ 6 ∗ 252a ⊕ 2 ∗ 320b ⊕ 2 ∗ 1242a ⊕ 1 ∗ 1792a
10*1a⊕7 ∗ 11b ⊕ 15 ∗ 11a ⊕ 8 ∗ 44a ⊕ 8 ∗ 44b ⊕ 6 ∗ 220b ⊕ 6 ∗ 120a ⊕ 1 ∗ 220a ⊕ 5 ∗ 252a ⊕ 3 ∗ 320a ⊕ 3 ∗ 1242a
10*1a⊕16 ∗ 11b ⊕ 9 ∗ 11a ⊕ 11 ∗ 44a ⊕ 8 ∗ 44b ⊕ 1 ∗ 220b ⊕ 6 ∗ 120a ⊕ 5 ∗ 220a ⊕ 3 ∗ 252a ⊕ 2 ∗ 320b ⊕ 3 ∗ 320a ⊕ 1 ∗ 1242a ⊕ 1 ∗ 1792a
9*1a⊕11 ∗ 11b ⊕ 13 ∗ 11a ⊕ 6 ∗ 44a ⊕ 8 ∗ 44b ⊕ 5 ∗ 220b ⊕ 11 ∗ 120a ⊕ 4 ∗ 220a ⊕ 3 ∗ 252a ⊕ 1 ∗ 320a ⊕ 2 ∗ 1242a ⊕ 1 ∗ 1792a
11*1a⊕12 ∗ 11b ⊕ 7 ∗ 11a ⊕ 8 ∗ 44a ⊕ 11 ∗ 44b ⊕ 3 ∗ 220b ⊕ 1 ∗ 120a ⊕ 4 ∗ 220a ⊕ 10 ∗ 252a ⊕ 3 ∗ 320b ⊕ 1 ∗ 320a
15*1a⊕15 ∗ 11b ⊕ 13 ∗ 11a ⊕ 7 ∗ 44a ⊕ 3 ∗ 44b ⊕ 3 ∗ 220b ⊕ 10 ∗ 120a ⊕ 2 ∗ 220a ⊕ 3 ∗ 252a ⊕ 2 ∗ 1242a
6*1a⊕5 ∗ 11b ⊕ 20 ∗ 11a ⊕ 12 ∗ 44a ⊕ 9 ∗ 44b ⊕ 5 ∗ 220b ⊕ 2 ∗ 120a ⊕ 4 ∗ 220a ⊕ 5 ∗ 252a ⊕ 2 ∗ 320b ⊕ 2 ∗ 320a ⊕ 1 ∗ 1242a
12*1a⊕16 ∗ 11b ⊕ 5 ∗ 11a ⊕ 4 ∗ 44a ⊕ 11 ∗ 44b ⊕ 4 ∗ 220b ⊕ 10 ∗ 120a ⊕ 9 ∗ 220a ⊕ 3 ∗ 320b ⊕ 1 ∗ 320a ⊕ 2 ∗ 1242a ⊕ 1 ∗ 1792a
9*1a⊕8 ∗ 11b ⊕ 14 ∗ 11a ⊕ 5 ∗ 44a ⊕ 7 ∗ 44b ⊕ 7 ∗ 220b ⊕ 2 ∗ 120a ⊕ 1 ∗ 220a ⊕ 7 ∗ 252a ⊕ 1 ∗ 320b ⊕ 3 ∗ 320a ⊕ 2 ∗ 1242a
11*1a⊕17 ∗ 11b ⊕ 8 ∗ 11a ⊕ 7 ∗ 44a ⊕ 5 ∗ 44b ⊕ 1 ∗ 220b ⊕ 8 ∗ 120a ⊕ 5 ∗ 220a ⊕ 6 ∗ 252a ⊕ 2 ∗ 320b ⊕ 2 ∗ 1242a ⊕ 1 ∗ 1792a
9*1a⊕4 ∗ 11b ⊕ 14 ∗ 11a ⊕ 7 ∗ 44a ⊕ 5 ∗ 44b ⊕ 2 ∗ 220b ⊕ 7 ∗ 120a ⊕ 4 ∗ 252a ⊕ 1 ∗ 320a ⊕ 2 ∗ 1242a
7*1a⊕5 ∗ 11b ⊕ 5 ∗ 11a ⊕ 9 ∗ 44a ⊕ 8 ∗ 44b ⊕ 1 ∗ 220b ⊕ 5 ∗ 120a ⊕ 4 ∗ 220a ⊕ 4 ∗ 252a ⊕ 2 ∗ 320b ⊕ 3 ∗ 320a ⊕ 1 ∗ 1792a
7*1a⊕10 ∗ 11b ⊕ 7 ∗ 11a ⊕ 6 ∗ 44a ⊕ 4 ∗ 44b ⊕ 6 ∗ 220b ⊕ 10 ∗ 120a ⊕ 1 ∗ 220a ⊕ 3 ∗ 252a ⊕ 1 ∗ 320b ⊕ 1 ∗ 1242a ⊕ 1 ∗ 1792a
7*1a⊕9 ∗ 11b ⊕ 8 ∗ 11a ⊕ 2 ∗ 44a ⊕ 10 ∗ 44b ⊕ 1 ∗ 120a ⊕ 4 ∗ 220a ⊕ 8 ∗ 252a ⊕ 2 ∗ 320b
9*1a⊕6 ∗ 11b ⊕ 11 ∗ 11a ⊕ 5 ∗ 44a ⊕ 3 ∗ 44b ⊕ 4 ∗ 220b ⊕ 4 ∗ 120a ⊕ 3 ∗ 220a ⊕ 2 ∗ 252a ⊕ 1 ∗ 1242a
3*1a⊕7 ∗ 11b ⊕ 6 ∗ 11a ⊕ 10 ∗ 44a ⊕ 7 ∗ 44b ⊕ 3 ∗ 220b ⊕ 2 ∗ 120a ⊕ 4 ∗ 220a ⊕ 2 ∗ 252a ⊕ 1 ∗ 320b ⊕ 2 ∗ 320a
3*1a⊕10 ∗ 11b ⊕ 6 ∗ 11a ⊕ 1 ∗ 44a ⊕ 5 ∗ 44b ⊕ 3 ∗ 220b ⊕ 7 ∗ 120a ⊕ 4 ∗ 220a ⊕ 2 ∗ 252a ⊕ 2 ∗ 320b ⊕ 2 ∗ 1242a
8*1a⊕5 ∗ 11b ⊕ 7 ∗ 11a ⊕ 3 ∗ 44a ⊕ 3 ∗ 44b ⊕ 3 ∗ 220b ⊕ 1 ∗ 120a ⊕ 4 ∗ 252a ⊕ 1 ∗ 320b ⊕ 1 ∗ 320a
6*1a⊕5 ∗ 11b ⊕ 5 ∗ 11a ⊕ 5 ∗ 44a ⊕ 2 ∗ 44b ⊕ 4 ∗ 120a ⊕ 2 ∗ 220a ⊕ 3 ∗ 252a ⊕ 1 ∗ 320a ⊕ 1 ∗ 1242a ⊕ 1 ∗ 1792a
1*1a⊕3 ∗ 11b ⊕ 7 ∗ 11a ⊕ 6 ∗ 44a ⊕ 2 ∗ 44b ⊕ 3 ∗ 220b ⊕ 4 ∗ 120a ⊕ 1 ∗ 220a ⊕ 1 ∗ 252a ⊕ 1 ∗ 320a ⊕ 1 ∗ 1242a
5*1a⊕5 ∗ 11b ⊕ 2 ∗ 11a ⊕ 3 ∗ 44a ⊕ 5 ∗ 44b ⊕ 1 ∗ 220b ⊕ 2 ∗ 120a ⊕ 3 ∗ 220a ⊕ 2 ∗ 252a ⊕ 2 ∗ 320b ⊕ 1 ∗ 320a ⊕ 1 ∗ 1792a
6*1a⊕5 ∗ 11b ⊕ 5 ∗ 11a ⊕ 1 ∗ 44a ⊕ 1 ∗ 44b ⊕ 2 ∗ 220b ⊕ 3 ∗ 120a ⊕ 1 ∗ 252a ⊕ 1 ∗ 1242a
2*1a⊕3 ∗ 11b ⊕ 5 ∗ 11a ⊕ 4 ∗ 44a ⊕ 1 ∗ 44b ⊕ 1 ∗ 220b ⊕ 1 ∗ 120a ⊕ 1 ∗ 220a ⊕ 5 ∗ 252a ⊕ 1 ∗ 320b
4*1a⊕2 ∗ 11b ⊕ 3 ∗ 11a ⊕ 1 ∗ 44a ⊕ 3 ∗ 44b ⊕ 1 ∗ 220b ⊕ 3 ∗ 120a ⊕ 2 ∗ 220a ⊕ 1 ∗ 320b ⊕ 1 ∗ 320a
2*1a⊕4 ∗ 11b ⊕ 4 ∗ 11a ⊕ 4 ∗ 44a ⊕ 3 ∗ 44b ⊕ 2 ∗ 220b ⊕ 1 ∗ 120a ⊕ 1 ∗ 220a ⊕ 1 ∗ 252a ⊕ 1 ∗ 320a
3*1a⊕5 ∗ 11b ⊕ 3 ∗ 11a ⊕ 1 ∗ 44a ⊕ 1 ∗ 44b ⊕ 1 ∗ 220b ⊕ 2 ∗ 120a ⊕ 1 ∗ 220a ⊕ 3 ∗ 252a ⊕ 2 ∗ 320b ⊕ 1 ∗ 1242a
4*1a⊕2 ∗ 11b ⊕ 4 ∗ 11a ⊕ 2 ∗ 44a ⊕ 3 ∗ 120a ⊕ 1 ∗ 252a
1*1a⊕2 ∗ 11a ⊕ 3 ∗ 44a ⊕ 3 ∗ 44b ⊕ 1 ∗ 220b ⊕ 1 ∗ 220a ⊕ 2 ∗ 252a ⊕ 1 ∗ 320a ⊕ 1 ∗ 1242a
2*1a⊕6 ∗ 11b ⊕ 1 ∗ 11a ⊕ 1 ∗ 44a ⊕ 1 ∗ 44b ⊕ 2 ∗ 220b ⊕ 3 ∗ 120a ⊕ 1 ∗ 220a ⊕ 1 ∗ 320b
2*1a⊕2 ∗ 11b ⊕ 5 ∗ 11a ⊕ 1 ∗ 44a ⊕ 2 ∗ 44b ⊕ 1 ∗ 220a ⊕ 3 ∗ 252a ⊕ 1 ∗ 320b
3*1a⊕1 ∗ 11b ⊕ 2 ∗ 11a ⊕ 1 ∗ 44a ⊕ 1 ∗ 44b ⊕ 1 ∗ 120a ⊕ 1 ∗ 220a ⊕ 1 ∗ 252a
1*1a⊕1 ∗ 11b ⊕ 2 ∗ 11a ⊕ 3 ∗ 44a ⊕ 1 ∗ 44b ⊕ 2 ∗ 220b ⊕ 2 ∗ 120a ⊕ 1 ∗ 252a ⊕ 1 ∗ 320a
5*11b⊕1 ∗ 11a ⊕ 3 ∗ 44b ⊕ 3 ∗ 120a ⊕ 2 ∗ 220a ⊕ 2 ∗ 252a ⊕ 1 ∗ 320b
5*1a⊕1 ∗ 11b ⊕ 3 ∗ 11a ⊕ 1 ∗ 44b ⊕ 1 ∗ 120a ⊕ 2 ∗ 252a
2*1a⊕1 ∗ 11b ⊕ 1 ∗ 11a ⊕ 2 ∗ 44a ⊕ 1 ∗ 44b ⊕ 1 ∗ 252a
2*11b⊕2 ∗ 11a ⊕ 2 ∗ 44a ⊕ 1 ∗ 220b ⊕ 3 ∗ 120a
1*11b⊕2 ∗ 11a ⊕ 3 ∗ 44b ⊕ 2 ∗ 220a ⊕ 2 ∗ 252a
4*1a⊕2 ∗ 11b ⊕ 1 ∗ 220b
1*11b⊕3 ∗ 11a ⊕ 2 ∗ 44a ⊕ 1 ∗ 220b ⊕ 2 ∗ 252a
1*11b⊕1 ∗ 11a ⊕ 2 ∗ 44b ⊕ 2 ∗ 120a ⊕ 2 ∗ 220a ⊕ 1 ∗ 320b
1*1a⊕1 ∗ 11a ⊕ 1 ∗ 44a ⊕ 1 ∗ 44b ⊕ 1 ∗ 220b ⊕ 1 ∗ 320a
2*11b⊕1 ∗ 44a ⊕ 1 ∗ 120a
1*11a⊕1 ∗ 252a
1*1a⊕1 ∗ 44a
1*11a⊕1 ∗ 44a ⊕ 1 ∗ 220b ⊕ 1 ∗ 120a
1*11b⊕1 ∗ 44b ⊕ 1 ∗ 220a ⊕ 1 ∗ 252a
1*1a

Socle length 65

FIGURE 5.2. Socle series of P(11a)

1 * 11a

1*11b⊕1 * 44a ⊕ 1 * 44b ⊕ 1 * 220a ⊕ 1 * 320b

1*11a⊕2 * 11b ⊕ 4 * 1a ⊕ 2 * 120a ⊕ 1 * 220b ⊕ 1 * 1242a

5*11a⊕1 * 11b ⊕ 1 * 1a ⊕ 1 * 44b ⊕ 1 * 120a ⊕ 4 * 252a ⊕ 1 * 320a ⊕ 1 * 1242a

2*11a⊕2 * 11b ⊕ 3 * 1a ⊕ 4 * 44a ⊕ 4 * 44b ⊕ 2 * 120a ⊕ 2 * 220a ⊕ 1 * 320a ⊕ 1 * 1242a ⊕ 1 * 1792a

3*11a⊕4 * 11b ⊕ 6 * 1a ⊕ 5 * 44a ⊕ 2 * 44b ⊕ 2 * 120a ⊕ 1 * 220a ⊕ 4 * 220b ⊕ 3 * 320a ⊕ 1 * 1792a

3*11a⊕13 * 11b ⊕ 6 * 1a ⊕ 2 * 44a ⊕ 5 * 44b ⊕ 2 * 120a ⊕ 2 * 220a ⊕ 1 * 220b ⊕ 3 * 252a ⊕ 3 * 320b ⊕ 1 * 1792a

11*11a⊕4 * 11b ⊕ 10 * 1a ⊕ 3 * 44a ⊕ 2 * 44b ⊕ 4 * 120a ⊕ 1 * 220b ⊕ 1 * 252a ⊕ 1 * 320b ⊕ 2 * 1242a

7*11a⊕2 * 11b ⊕ 3 * 1a ⊕ 8 * 44a ⊕ 5 * 44b ⊕ 3 * 120a ⊕ 3 * 220a ⊕ 1 * 220b ⊕ 4 * 252a ⊕ 3 * 320a ⊕ 1 * 1242a

5*11a⊕13 * 11b ⊕ 9 * 1a ⊕ 3 * 44a ⊕ 3 * 44b ⊕ 7 * 120a ⊕ 3 * 220a ⊕ 4 * 220b ⊕ 1 * 320b ⊕ 1 * 320a ⊕ 1 * 1242a ⊕ 1 * 1792a

10*11a⊕6 * 11b ⊕ 6 * 1a ⊕ 3 * 44a ⊕ 9 * 44b ⊕ 4 * 120a ⊕ 3 * 220a ⊕ 2 * 220b ⊕ 6 * 252a ⊕ 1 * 320b ⊕ 1 * 320a ⊕ 1 * 1242a ⊕ 1 * 1792a

8*11a⊕11 * 11b ⊕ 11 * 1a ⊕ 6 * 44a ⊕ 6 * 44b ⊕ 4 * 120a ⊕ 2 * 220a ⊕ 2 * 220b ⊕ 3 * 252a ⊕ 1 * 320b ⊕ 1 * 320a ⊕ 2 * 1242a

14*11a⊕7 * 11b ⊕ 7 * 1a ⊕ 10 * 44a ⊕ 3 * 44b ⊕ 4 * 120a ⊕ 5 * 220b ⊕ 4 * 252a ⊕ 3 * 320a ⊕ 1 * 1792a

7*11a⊕15 * 11b ⊕ 10 * 1a ⊕ 7 * 44a ⊕ 11 * 44b ⊕ 7 * 120a ⊕ 8 * 220a ⊕ 3 * 252a ⊕ 4 * 320b ⊕ 2 * 1242a ⊕ 2 * 1792a

14*11a⊕6 * 11b ⊕ 16 * 1a ⊕ 7 * 44a ⊕ 7 * 44b ⊕ 7 * 120a ⊕ 1 * 220a ⊕ 6 * 220b ⊕ 3 * 252a ⊕ 1 * 320b ⊕ 1 * 320a ⊕ 3 * 1242a

7*11a⊕11 * 11b ⊕ 8 * 1a ⊕ 9 * 44a ⊕ 6 * 44b ⊕ 3 * 120a ⊕ 4 * 220a ⊕ 2 * 220b ⊕ 9 * 252a ⊕ 1 * 320b ⊕ 4 * 320a

13*11a⊕15 * 11b ⊕ 13 * 1a ⊕ 4 * 44a ⊕ 7 * 44b ⊕ 14 * 120a ⊕ 3 * 220a ⊕ 6 * 220b ⊕ 1 * 252a ⊕ 1 * 320b ⊕ 3 * 1242a ⊕ 2 * 1792a

9*11a⊕12 * 11b ⊕ 9 * 1a ⊕ 9 * 44a ⊕ 18 * 44b ⊕ 1 * 120a ⊕ 6 * 220a ⊕ 4 * 220b ⊕ 6 * 252a ⊕ 3 * 320b ⊕ 4 * 320a ⊕ 1 * 1242a

11*11a⊕24 * 11b ⊕ 15 * 1a ⊕ 8 * 44a ⊕ 2 * 44b ⊕ 10 * 120a ⊕ 2 * 220a ⊕ 3 * 220b ⊕ 3 * 252a ⊕ 2 * 1242a ⊕ 1 * 1792a

25*11a⊕3 * 11b ⊕ 7 * 1a ⊕ 11 * 44a ⊕ 6 * 44b ⊕ 5 * 120a ⊕ 2 * 220a ⊕ 2 * 220b ⊕ 10 * 252a ⊕ 1 * 320b ⊕ 1 * 320a ⊕ 2 * 1242a

3*11a⊕9 * 11b ⊕ 16 * 1a ⊕ 6 * 44a ⊕ 16 * 44b ⊕ 11 * 120a ⊕ 11 * 220a ⊕ 2 * 220b ⊕ 1 * 252a ⊕ 3 * 320b ⊕ 2 * 320a ⊕ 2 * 1242a ⊕ 1 * 1792a

12*11a⊕15 * 11b ⊕ 11 * 1a ⊕ 10 * 44a ⊕ 8 * 44b ⊕ 4 * 120a ⊕ 13 * 220b ⊕ 6 * 252a ⊕ 4 * 320a ⊕ 1 * 1242a

12*11a⊕27 * 11b ⊕ 12 * 1a ⊕ 5 * 44a ⊕ 7 * 44b ⊕ 5 * 120a ⊕ 6 * 220a ⊕ 9 * 252a ⊕ 4 * 320b ⊕ 1 * 1792a

24*11a⊕3 * 11b ⊕ 15 * 1a ⊕ 7 * 44a ⊕ 5 * 44b ⊕ 11 * 120a ⊕ 3 * 220a ⊕ 3 * 220b ⊕ 3 * 252a ⊕ 6 * 1242a

6*11a⊕7 * 11b ⊕ 5 * 1a ⊕ 17 * 44a ⊕ 12 * 44b ⊕ 2 * 120a ⊕ 6 * 220a ⊕ 4 * 220b ⊕ 7 * 252a ⊕ 2 * 320b ⊕ 4 * 320a

9*11a⊕18 * 11b ⊕ 16 * 1a ⊕ 5 * 44a ⊕ 3 * 44b ⊕ 18 * 120a ⊕ 2 * 220a ⊕ 8 * 220b ⊕ 1 * 252a ⊕ 1 * 320b ⊕ 2 * 1242a ⊕ 2 * 1792a

15*11a⊕9 * 11b ⊕ 8 * 1a ⊕ 4 * 44a ⊕ 15 * 44b ⊕ 1 * 120a ⊕ 5 * 220a ⊕ 1 * 220b ⊕ 10 * 252a ⊕ 3 * 320b ⊕ 3 * 320a ⊕ 1 * 1242a

9*11a⊕14 * 11b ⊕ 17 * 1a ⊕ 6 * 44a ⊕ 4 * 44b ⊕ 7 * 120a ⊕ 3 * 220a ⊕ 3 * 220b ⊕ 2 * 252a ⊕ 2 * 1242a ⊕ 1 * 1792a

15*11a⊕5 * 11b ⊕ 4 * 1a ⊕ 15 * 44a ⊕ 4 * 44b ⊕ 4 * 120a ⊕ 6 * 220b ⊕ 8 * 252a ⊕ 1 * 320b ⊕ 2 * 320a

5*11a⊕13 * 11b ⊕ 5 * 1a ⊕ 4 * 44a ⊕ 13 * 44b ⊕ 13 * 120a ⊕ 10 * 220a ⊕ 2 * 252a ⊕ 3 * 320b ⊕ 2 * 1242a ⊕ 1 * 1792a

13*11a⊕6 * 11b ⊕ 10 * 1a ⊕ 7 * 44a ⊕ 7 * 44b ⊕ 2 * 120a ⊕ 8 * 220b ⊕ 4 * 252a ⊕ 2 * 320a

6*11a⊕16 * 11b ⊕ 9 * 1a ⊕ 6 * 44a ⊕ 5 * 44b ⊕ 3 * 120a ⊕ 4 * 220a ⊕ 5 * 252a ⊕ 3 * 320b ⊕ 1 * 320a ⊕ 1 * 1792a

14*11a⊕6 * 11b ⊕ 6 * 1a ⊕ 4 * 44a ⊕ 5 * 44b ⊕ 8 * 120a ⊕ 3 * 220a ⊕ 5 * 220b ⊕ 3 * 252a ⊕ 3 * 1242a

5*11a⊕9 * 11b ⊕ 7 * 1a ⊕ 9 * 44a ⊕ 10 * 44b ⊕ 6 * 220a ⊕ 3 * 220b ⊕ 5 * 252a ⊕ 1 * 320b ⊕ 2 * 320a

11*11a⊕11 * 11b ⊕ 10 * 1a ⊕ 3 * 44a ⊕ 7 * 120a ⊕ 1 * 220a ⊕ 3 * 220b ⊕ 1 * 252a ⊕ 1 * 320b ⊕ 1 * 1242a ⊕ 1 * 1792a

8*11a⊕3 * 11b ⊕ 5 * 1a ⊕ 8 * 44a ⊕ 5 * 44b ⊕ 2 * 120a ⊕ 2 * 220a ⊕ 2 * 220b ⊕ 6 * 252a ⊕ 3 * 320b ⊕ 1 * 1242a

3*11a⊕6 * 11b ⊕ 5 * 1a ⊕ 2 * 44a ⊕ 4 * 44b ⊕ 7 * 120a ⊕ 3 * 220a ⊕ 2 * 220b ⊕ 1 * 320b ⊕ 2 * 320a ⊕ 2 * 1242a

5*11a⊕6 * 11b ⊕ 3 * 1a ⊕ 4 * 44a ⊕ 3 * 44b ⊕ 2 * 120a ⊕ 3 * 220b ⊕ 3 * 252a ⊕ 2 * 320a ⊕ 1 * 1792a

5*11a⊕7 * 11b ⊕ 5 * 1a ⊕ 1 * 44a ⊕ 3 * 44b ⊕ 3 * 120a ⊕ 2 * 220a ⊕ 4 * 252a ⊕ 1 * 320b ⊕ 1 * 1242a ⊕ 1 * 1792a

7*11a⊕2 * 11b ⊕ 5 * 1a ⊕ 4 * 44a ⊕ 1 * 44b ⊕ 4 * 120a ⊕ 1 * 220b ⊕ 1 * 252a

2*11a⊕2 * 11b ⊕ 3 * 1a ⊕ 4 * 44a ⊕ 5 * 44b ⊕ 2 * 120a ⊕ 2 * 220a ⊕ 1 * 220b ⊕ 2 * 252a ⊕ 1 * 320b ⊕ 2 * 320a

3*11a⊕8 * 11b ⊕ 2 * 1a ⊕ 1 * 44a ⊕ 2 * 44b ⊕ 4 * 120a ⊕ 1 * 220a ⊕ 3 * 220b ⊕ 1 * 252a ⊕ 1 * 320b ⊕ 1 * 1242a

6*11a⊕2 * 11b ⊕ 4 * 1a ⊕ 2 * 44a ⊕ 3 * 44b ⊕ 2 * 220a ⊕ 4 * 252a ⊕ 1 * 320b

3*11a⊕1 * 11b ⊕ 5 * 1a ⊕ 2 * 44a ⊕ 1 * 44b ⊕ 1 * 120a ⊕ 1 * 220a ⊕ 1 * 220b ⊕ 1 * 252a

3*11a⊕3 * 11b ⊕ 2 * 1a ⊕ 5 * 44a ⊕ 1 * 44b ⊕ 2 * 120a ⊕ 3 * 220b ⊕ 1 * 252a ⊕ 1 * 320b ⊕ 1 * 320a

3*11a⊕6 * 11b ⊕ 4 * 44b ⊕ 3 * 120a ⊕ 3 * 220a ⊕ 2 * 252a ⊕ 2 * 320b ⊕ 1 * 1242a

5*11a⊕2 * 11b ⊕ 6 * 1a ⊕ 1 * 44a ⊕ 1 * 44b ⊕ 1 * 120a ⊕ 1 * 220b ⊕ 2 * 252a

2*11a⊕2 * 11b ⊕ 2 * 1a ⊕ 4 * 44a ⊕ 1 * 44b ⊕ 3 * 252a ⊕ 1 * 320a

3*11a⊕3 * 11b ⊕ 1 * 1a ⊕ 2 * 44a ⊕ 1 * 44b ⊕ 6 * 120a ⊕ 1 * 220a ⊕ 1 * 220b

2*11a⊕2 * 11b ⊕ 1 * 1a ⊕ 1 * 44a ⊕ 5 * 44b ⊕ 2 * 220a ⊕ 1 * 220b ⊕ 3 * 252a ⊕ 1 * 320b ⊕ 1 * 320a

2*11a⊕6 * 11b ⊕ 5 * 1a ⊕ 1 * 120a ⊕ 1 * 220b ⊕ 1 * 252a

6*11a⊕1 * 11b ⊕ 1 * 1a ⊕ 3 * 44a ⊕ 1 * 220b ⊕ 3 * 252a

1*11a⊕2 * 11b ⊕ 1 * 1a ⊕ 1 * 44a ⊕ 3 * 44b ⊕ 3 * 120a ⊕ 3 * 220a ⊕ 1 * 320b

3*11a⊕2 * 1a ⊕ 2 * 44a ⊕ 2 * 44b ⊕ 1 * 120a ⊕ 2 * 220b ⊕ 1 * 252a ⊕ 1 * 320a

4*11b⊕1 * 1a ⊕ 1 * 44a ⊕ 1 * 44b ⊕ 1 * 120a ⊕ 1 * 220a ⊕ 2 * 252a ⊕ 1 * 320b

3*11a⊕2 * 1a ⊕ 1 * 44a ⊕ 1 * 120a ⊕ 1 * 252a

1*11a⊕1 * 1a ⊕ 3 * 44a ⊕ 2 * 44b ⊕ 1 * 220a ⊕ 1 * 220b ⊕ 1 * 252a

1*11a⊕4 * 11b ⊕ 1 * 1a ⊕ 1 * 44a ⊕ 3 * 120a ⊕ 2 * 220b ⊕ 1 * 320b

3*11a⊕2 * 11b ⊕ 2 * 44b ⊕ 1 * 220a ⊕ 1 * 252a

2*11a⊕1 * 11b ⊕ 2 * 1a ⊕ 1 * 44a ⊕ 1 * 220a

1*11a⊕2 * 44a ⊕ 1 * 120a ⊕ 1 * 252a

2*120a⊕1 * 220a

1*44b⊕1 * 220b ⊕ 1 * 252a

1*11b⊕1 * 1a

1*11a

Socle length 65

FIGURE 5.3. Socle series of P(11b)

1*11b
1*11a⊕1 ∗ 1a ⊕ 1 ∗ 120a ⊕ 1 ∗ 220b ⊕ 1 ∗ 1242a
1*11b⊕2 ∗ 11a ⊕ 1 ∗ 44a ⊕ 3 ∗ 44b ⊕ 1 ∗ 220a ⊕ 1 ∗ 252a ⊕ 2 ∗ 320a
5*11b⊕1 ∗ 11a ⊕ 5 ∗ 1a ⊕ 2 ∗ 44a ⊕ 2 ∗ 120a ⊕ 1 ∗ 220b ⊕ 2 ∗ 1792a
2*11b⊕4 ∗ 11a ⊕ 3 ∗ 1a ⊕ 1 ∗ 44a ⊕ 2 ∗ 44b ⊕ 1 ∗ 120a ⊕ 3 ∗ 252a ⊕ 2 ∗ 320b ⊕ 1 ∗ 320a ⊕ 1 ∗ 1242a
3*11b⊕3 ∗ 11a ⊕ 3 ∗ 1a ⊕ 4 ∗ 44a ⊕ 3 ∗ 44b ⊕ 4 ∗ 120a ⊕ 2 ∗ 220a ⊕ 1 ∗ 252a ⊕ 1 ∗ 320a ⊕ 2 ∗ 1242a
3*11b⊕6 ∗ 11a ⊕ 7 ∗ 1a ⊕ 5 ∗ 44a ⊕ 2 ∗ 44b ⊕ 3 ∗ 120a ⊕ 1 ∗ 220a ⊕ 3 ∗ 220b ⊕ 2 ∗ 320a ⊕ 1 ∗ 1792a
11*11b⊕3 ∗ 11a ⊕ 5 ∗ 1a ⊕ 2 ∗ 44a ⊕ 7 ∗ 44b ⊕ 3 ∗ 120a ⊕ 4 ∗ 220a ⊕ 1 ∗ 220b ⊕ 3 ∗ 252a ⊕ 2 ∗ 320b ⊕ 1 ∗ 320a ⊕ 1 ∗ 1792a
7*11b⊕10 ∗ 11a ⊕ 9 ∗ 1a ⊕ 3 ∗ 44a ⊕ 3 ∗ 44b ⊕ 3 ∗ 120a ⊕ 3 ∗ 220b ⊕ 2 ∗ 252a ⊕ 1 ∗ 320b ⊕ 1 ∗ 1242a
5*11b⊕7 ∗ 11a ⊕ 6 ∗ 1a ⊕ 7 ∗ 44a ⊕ 5 ∗ 44b ⊕ 3 ∗ 120a ⊕ 2 ∗ 220a ⊕ 1 ∗ 220b ⊕ 5 ∗ 252a ⊕ 2 ∗ 320a ⊕ 1 ∗ 1242a
10*11b⊕10 ∗ 11a ⊕ 7 ∗ 1a ⊕ 5 ∗ 44a ⊕ 4 ∗ 44b ⊕ 7 ∗ 120a ⊕ 3 ∗ 220a ⊕ 4 ∗ 220b ⊕ 1 ∗ 320b ⊕ 1 ∗ 320a ⊕ 2 ∗ 1242a ⊕ 1 ∗ 1792a
8*11b⊕7 ∗ 11a ⊕ 8 ∗ 1a ⊕ 6 ∗ 44a ⊕ 10 ∗ 44b ⊕ 4 ∗ 120a ⊕ 5 ∗ 220a ⊕ 2 ∗ 220b ⊕ 3 ∗ 252a ⊕ 2 ∗ 320b ⊕ 3 ∗ 320a ⊕ 1 ∗ 1242a ⊕ 1 ∗ 1792a
14*11b⊕9 ∗ 11a ⊕ 12 ∗ 1a ⊕ 5 ∗ 44a ⊕ 5 ∗ 44b ⊕ 5 ∗ 120a ⊕ 2 ∗ 220a ⊕ 3 ∗ 220b ⊕ 3 ∗ 252a ⊕ 1 ∗ 320b ⊕ 1 ∗ 320a ⊕ 2 ∗ 1242a ⊕ 1 ∗ 1792a
7*11b⊕13 ∗ 11a ⊕ 9 ∗ 1a ⊕ 7 ∗ 44a ⊕ 3 ∗ 44b ⊕ 4 ∗ 120a ⊕ 4 ∗ 220b ⊕ 6 ∗ 252a ⊕ 1 ∗ 320b ⊕ 2 ∗ 320a ⊕ 1 ∗ 1242a ⊕ 1 ∗ 1792a
14*11b⊕6 ∗ 11a ⊕ 9 ∗ 1a ⊕ 6 ∗ 44a ⊕ 10 ∗ 44b ⊕ 7 ∗ 120a ⊕ 7 ∗ 220a ⊕ 3 ∗ 252a ⊕ 2 ∗ 320b ⊕ 1 ∗ 320a ⊕ 2 ∗ 1242a ⊕ 1 ∗ 1792a
7*11b⊕14 ∗ 11a ⊕ 15 ∗ 1a ⊕ 9 ∗ 44a ⊕ 7 ∗ 44b ⊕ 6 ∗ 120a ⊕ 1 ∗ 220a ⊕ 7 ∗ 220b ⊕ 3 ∗ 252a ⊕ 1 ∗ 320a ⊕ 2 ∗ 1242a ⊕ 1 ∗ 1792a
13*11b⊕8 ∗ 11a ⊕ 8 ∗ 1a ⊕ 8 ∗ 44a ⊕ 8 ∗ 44b ⊕ 4 ∗ 120a ⊕ 4 ∗ 220a ⊕ 1 ∗ 220b ⊕ 9 ∗ 252a ⊕ 2 ∗ 320b ⊕ 2 ∗ 320a
9*11b⊕16 ∗ 11a ⊕ 12 ∗ 1a ⊕ 6 ∗ 44a ⊕ 6 ∗ 44b ⊕ 13 ∗ 120a ⊕ 3 ∗ 220a ⊕ 5 ∗ 220b ⊕ 1 ∗ 252a ⊕ 1 ∗ 320b ⊕ 4 ∗ 1242a ⊕ 1 ∗ 1792a
11*11b⊕9 ∗ 11a ⊕ 8 ∗ 1a ⊕ 11 ∗ 44a ⊕ 14 ∗ 44b ⊕ 1 ∗ 120a ⊕ 5 ∗ 220a ⊕ 5 ∗ 220b ⊕ 6 ∗ 252a ⊕ 3 ∗ 320b ⊕ 5 ∗ 320a ⊕ 1 ∗ 1242a
25*11b⊕11 ∗ 11a ⊕ 15 ∗ 1a ⊕ 5 ∗ 44a ⊕ 4 ∗ 44b ⊕ 12 ∗ 120a ⊕ 4 ∗ 220a ⊕ 3 ∗ 220b ⊕ 3 ∗ 252a ⊕ 1 ∗ 320b ⊕ 2 ∗ 1242a ⊕ 2 ∗ 1792a
3*11b⊕19 ∗ 11a ⊕ 9 ∗ 1a ⊕ 8 ∗ 44a ⊕ 8 ∗ 44b ⊕ 6 ∗ 120a ⊕ 2 ∗ 220a ⊕ 3 ∗ 220b ⊕ 11 ∗ 252a ⊕ 1 ∗ 320b ⊕ 1 ∗ 320a ⊕ 2 ∗ 1242a
12*11b⊕4 ∗ 11a ⊕ 13 ∗ 1a ⊕ 8 ∗ 44a ⊕ 12 ∗ 44b ⊕ 11 ∗ 120a ⊕ 7 ∗ 220a ⊕ 2 ∗ 220b ⊕ 1 ∗ 252a ⊕ 1 ∗ 320b ⊕ 3 ∗ 320a ⊕ 2 ∗ 1242a
12*11b⊕17 ∗ 11a ⊕ 7 ∗ 1a ⊕ 11 ∗ 44a ⊕ 8 ∗ 44b ⊕ 3 ∗ 120a ⊕ 10 ∗ 220b ⊕ 6 ∗ 252a ⊕ 4 ∗ 320a ⊕ 1 ∗ 1242a ⊕ 1 ∗ 1792a
24*11b⊕8 ∗ 11a ⊕ 13 ∗ 1a ⊕ 6 ∗ 44a ⊕ 12 ∗ 44b ⊕ 4 ∗ 120a ⊕ 10 ∗ 220a ⊕ 6 ∗ 252a ⊕ 5 ∗ 320b ⊕ 1 ∗ 1792a
6*11b⊕21 ∗ 11a ⊕ 20 ∗ 1a ⊕ 8 ∗ 44a ⊕ 4 ∗ 44b ⊕ 8 ∗ 120a ⊕ 2 ∗ 220a ⊕ 6 ∗ 220b ⊕ 3 ∗ 252a ⊕ 3 ∗ 1242a
9*11b⊕9 ∗ 11a ⊕ 5 ∗ 1a ⊕ 15 ∗ 44a ⊕ 10 ∗ 44b ⊕ 2 ∗ 120a ⊕ 6 ∗ 220a ⊕ 3 ∗ 220b ⊕ 9 ∗ 252a ⊕ 2 ∗ 320b ⊕ 2 ∗ 320a
15*11b⊕12 ∗ 11a ⊕ 14 ∗ 1a ⊕ 3 ∗ 44a ⊕ 5 ∗ 44b ⊕ 15 ∗ 120a ⊕ 3 ∗ 220a ⊕ 8 ∗ 220b ⊕ 1 ∗ 252a ⊕ 2 ∗ 320b ⊕ 4 ∗ 1242a ⊕ 1 ∗ 1792a
9*11b⊕10 ∗ 11a ⊕ 8 ∗ 1a ⊕ 7 ∗ 44a ⊕ 12 ∗ 44b ⊕ 2 ∗ 120a ⊕ 5 ∗ 220a ⊕ 2 ∗ 220b ⊕ 6 ∗ 252a ⊕ 3 ∗ 320b ⊕ 4 ∗ 320a ⊕ 1 ∗ 1242a
15*11b⊕9 ∗ 11a ⊕ 14 ∗ 1a ⊕ 6 ∗ 44a ⊕ 2 ∗ 44b ⊕ 11 ∗ 120a ⊕ 1 ∗ 220a ⊕ 3 ∗ 220b ⊕ 2 ∗ 252a ⊕ 1 ∗ 1242a ⊕ 1 ∗ 1792a
5*11b⊕13 ∗ 11a ⊕ 5 ∗ 1a ⊕ 11 ∗ 44a ⊕ 7 ∗ 44b ⊕ 4 ∗ 120a ⊕ 3 ∗ 220b ⊕ 11 ∗ 252a ⊕ 1 ∗ 320b ⊕ 2 ∗ 320a ⊕ 1 ∗ 1242a
13*11b⊕5 ∗ 11a ⊕ 7 ∗ 1a ⊕ 5 ∗ 44a ⊕ 11 ∗ 44b ⊕ 13 ∗ 120a ⊕ 8 ∗ 220a ⊕ 2 ∗ 252a ⊕ 1 ∗ 320b ⊕ 2 ∗ 1242a ⊕ 1 ∗ 1792a
6*11b⊕16 ∗ 11a ⊕ 8 ∗ 1a ⊕ 8 ∗ 44a ⊕ 6 ∗ 44b ⊕ 3 ∗ 120a ⊕ 7 ∗ 220b ⊕ 5 ∗ 252a ⊕ 2 ∗ 320a
14*11b⊕5 ∗ 11a ⊕ 11 ∗ 1a ⊕ 4 ∗ 44a ⊕ 8 ∗ 44b ⊕ 3 ∗ 120a ⊕ 7 ∗ 220a ⊕ 4 ∗ 252a ⊕ 4 ∗ 320b ⊕ 1 ∗ 320a ⊕ 1 ∗ 1792a
5*11b⊕13 ∗ 11a ⊕ 6 ∗ 1a ⊕ 4 ∗ 44a ⊕ 4 ∗ 44b ⊕ 5 ∗ 120a ⊕ 2 ∗ 220a ⊕ 8 ∗ 220b ⊕ 3 ∗ 252a ⊕ 2 ∗ 1242a
11*11b⊕3 ∗ 11a ⊕ 6 ∗ 1a ⊕ 8 ∗ 44a ⊕ 8 ∗ 44b ⊕ 6 ∗ 220a ⊕ 2 ∗ 220b ⊕ 5 ∗ 252a ⊕ 2 ∗ 320b ⊕ 2 ∗ 320a
8*11b⊕13 ∗ 11a ⊕ 7 ∗ 1a ⊕ 3 ∗ 44a ⊕ 8 ∗ 120a ⊕ 3 ∗ 220b ⊕ 1 ∗ 252a ⊕ 2 ∗ 1242a ⊕ 1 ∗ 1792a
3*11b⊕8 ∗ 11a ⊕ 5 ∗ 1a ⊕ 9 ∗ 44a ⊕ 6 ∗ 44b ⊕ 1 ∗ 120a ⊕ 2 ∗ 220a ⊕ 1 ∗ 220b ⊕ 6 ∗ 252a ⊕ 3 ∗ 320b ⊕ 1 ∗ 320a
5*11b⊕3 ∗ 11a ⊕ 7 ∗ 1a ⊕ 2 ∗ 44a ⊕ 3 ∗ 44b ⊕ 7 ∗ 120a ⊕ 2 ∗ 220a ⊕ 2 ∗ 220b ⊕ 1 ∗ 320b ⊕ 1 ∗ 320a ⊕ 2 ∗ 1242a ⊕ 1 ∗ 1792a
5*11b⊕5 ∗ 11a ⊕ 2 ∗ 1a ⊕ 5 ∗ 44a ⊕ 4 ∗ 44b ⊕ 1 ∗ 120a ⊕ 2 ∗ 220b ⊕ 4 ∗ 252a ⊕ 1 ∗ 320a
7*11b⊕4 ∗ 11a ⊕ 5 ∗ 1a ⊕ 1 ∗ 44a ⊕ 3 ∗ 44b ⊕ 3 ∗ 120a ⊕ 3 ∗ 220a ⊕ 3 ∗ 252a ⊕ 1 ∗ 320b ⊕ 1 ∗ 1242a
2*11b⊕8 ∗ 11a ⊕ 5 ∗ 1a ⊕ 5 ∗ 44a ⊕ 1 ∗ 44b ⊕ 3 ∗ 120a ⊕ 2 ∗ 220b ⊕ 1 ∗ 252a
3*11b⊕3 ∗ 11a ⊕ 3 ∗ 1a ⊕ 3 ∗ 44a ⊕ 4 ∗ 44b ⊕ 2 ∗ 120a ⊕ 3 ∗ 220a ⊕ 1 ∗ 220b ⊕ 3 ∗ 252a ⊕ 2 ∗ 320b ⊕ 1 ∗ 320a
6*11b⊕4 ∗ 11a ⊕ 4 ∗ 1a ⊕ 1 ∗ 44a ⊕ 1 ∗ 44b ⊕ 4 ∗ 120a ⊕ 1 ∗ 220a ⊕ 4 ∗ 220b ⊕ 1 ∗ 252a ⊕ 1 ∗ 320b ⊕ 1 ∗ 1242a
3*11b⊕5 ∗ 11a ⊕ 3 ∗ 1a ⊕ 3 ∗ 44a ⊕ 4 ∗ 44b ⊕ 2 ∗ 220a ⊕ 3 ∗ 252a ⊕ 1 ∗ 320b ⊕ 1 ∗ 320a
3*11b⊕2 ∗ 11a ⊕ 4 ∗ 1a ⊕ 2 ∗ 44a ⊕ 1 ∗ 44b ⊕ 2 ∗ 120a ⊕ 1 ∗ 220a ⊕ 1 ∗ 220b ⊕ 1 ∗ 252a ⊕ 1 ∗ 1242a
3*11b⊕3 ∗ 11a ⊕ 2 ∗ 1a ⊕ 4 ∗ 44a ⊕ 1 ∗ 44b ⊕ 2 ∗ 120a ⊕ 2 ∗ 220b ⊕ 2 ∗ 252a ⊕ 1 ∗ 320a
5*11b⊕3 ∗ 11a ⊕ 1 ∗ 1a ⊕ 3 ∗ 44b ⊕ 3 ∗ 120a ⊕ 2 ∗ 220a ⊕ 2 ∗ 252a ⊕ 1 ∗ 320b
2*11b⊕5 ∗ 11a ⊕ 5 ∗ 1a ⊕ 2 ∗ 44a ⊕ 1 ∗ 44b ⊕ 1 ∗ 120a ⊕ 2 ∗ 220b ⊕ 2 ∗ 252a
3*11b⊕2 ∗ 11a ⊕ 2 ∗ 1a ⊕ 3 ∗ 44a ⊕ 2 ∗ 44b ⊕ 1 ∗ 220a ⊕ 2 ∗ 252a ⊕ 1 ∗ 320b ⊕ 1 ∗ 320a
2*11b⊕4 ∗ 11a ⊕ 2 ∗ 1a ⊕ 2 ∗ 44a ⊕ 1 ∗ 44b ⊕ 4 ∗ 120a ⊕ 1 ∗ 220a ⊕ 1 ∗ 220b
2*11b⊕1 ∗ 1a ⊕ 2 ∗ 44a ⊕ 4 ∗ 44b ⊕ 2 ∗ 220a ⊕ 1 ∗ 220b ⊕ 2 ∗ 252a ⊕ 1 ∗ 320b ⊕ 1 ∗ 320a
6*11b⊕1 ∗ 11a ⊕ 3 ∗ 1a ⊕ 3 ∗ 120a ⊕ 1 ∗ 220b ⊕ 1 ∗ 252a
1*11b⊕6 ∗ 11a ⊕ 1 ∗ 1a ⊕ 2 ∗ 44a ⊕ 1 ∗ 44b ⊕ 1 ∗ 220b ⊕ 4 ∗ 252a
3*11b⊕1 ∗ 11a ⊕ 2 ∗ 1a ⊕ 1 ∗ 44a ⊕ 2 ∗ 44b ⊕ 3 ∗ 120a ⊕ 2 ∗ 220a
4*11a⊕2 ∗ 1a ⊕ 3 ∗ 44a ⊕ 2 ∗ 44b ⊕ 1 ∗ 120a ⊕ 1 ∗ 220b ⊕ 1 ∗ 320a
3*11b⊕2 ∗ 44b ⊕ 1 ∗ 120a ⊕ 2 ∗ 220a ⊕ 2 ∗ 252a ⊕ 1 ∗ 320b
1*11b⊕2 ∗ 11a ⊕ 3 ∗ 1a ⊕ 1 ∗ 120a ⊕ 1 ∗ 220b ⊕ 1 ∗ 252a
1*11b⊕1 ∗ 1a ⊕ 2 ∗ 44a ⊕ 2 ∗ 44b ⊕ 1 ∗ 220a ⊕ 2 ∗ 252a
3*11b⊕2 ∗ 11a ⊕ 1 ∗ 1a ⊕ 1 ∗ 44a ⊕ 3 ∗ 120a ⊕ 2 ∗ 220b
2*11b⊕1 ∗ 11a ⊕ 1 ∗ 44a ⊕ 2 ∗ 44b ⊕ 1 ∗ 220a ⊕ 1 ∗ 320a
1*11b⊕2 ∗ 11a ⊕ 1 ∗ 1a ⊕ 1 ∗ 120a
1*11a⊕2 ∗ 252a
1*1a⊕1 ∗ 120a ⊕ 1 ∗ 220a
1*44a⊕1 ∗ 44b ⊕ 1 ∗ 220b
1*11b

Socle length 65

FIGURE 5.4. Socle series of P(44a)

1*44a

1*11b⊕1 * 1a ⊕ 1 * 120a ⊕ 1 * 220a

3*11a⊕1 * 1a ⊕ 1 * 44b ⊕ 1 * 220b ⊕ 1 * 252a ⊕ 1 * 1242a

1*11b⊕1 * 1a ⊕ 3 * 44a ⊕ 2 * 44b ⊕ 1 * 252a ⊕ 1 * 320a

4*11b⊕2 * 11a ⊕ 5 * 1a ⊕ 1 * 44a ⊕ 3 * 120a ⊕ 1 * 220b ⊕ 1 * 1792a

2*11b⊕3 * 11a ⊕ 1 * 1a ⊕ 1 * 44a ⊕ 5 * 44b ⊕ 2 * 220a ⊕ 1 * 252a ⊕ 2 * 320b ⊕ 1 * 320a ⊕ 1 * 1242a

5*11b⊕2 * 11a ⊕ 5 * 1a ⊕ 5 * 44a ⊕ 2 * 120a ⊕ 1 * 220a ⊕ 1 * 220b ⊕ 1 * 1242a

2*11b⊕7 * 11a ⊕ 5 * 1a ⊕ 2 * 44a ⊕ 3 * 120a ⊕ 2 * 220b ⊕ 3 * 252a ⊕ 1 * 320a

5*11b⊕3 * 11a ⊕ 5 * 1a ⊕ 7 * 44b ⊕ 3 * 120a ⊕ 3 * 220a ⊕ 2 * 252a ⊕ 1 * 320b ⊕ 1 * 1242a ⊕ 1 * 1792a

3*11b⊕5 * 11a ⊕ 5 * 1a ⊕ 5 * 44a ⊕ 3 * 44b ⊕ 2 * 120a ⊕ 1 * 220a ⊕ 3 * 220b ⊕ 1 * 252a ⊕ 2 * 320a ⊕ 1 * 1242a

9*11b⊕4 * 11a ⊕ 4 * 1a ⊕ 6 * 44a ⊕ 2 * 44b ⊕ 2 * 120a ⊕ 2 * 220b ⊕ 4 * 252a ⊕ 1 * 320a ⊕ 1 * 1792a

6*11b⊕10 * 11a ⊕ 8 * 1a ⊕ 2 * 44a ⊕ 4 * 44b ⊕ 5 * 120a ⊕ 1 * 220a ⊕ 1 * 220b ⊕ 1 * 252a ⊕ 2 * 320b ⊕ 1 * 1242a ⊕ 1 * 1792a

3*11b⊕5 * 11a ⊕ 4 * 1a ⊕ 7 * 44a ⊕ 9 * 44b ⊕ 1 * 120a ⊕ 5 * 220a ⊕ 1 * 220b ⊕ 1 * 252a ⊕ 1 * 320b ⊕ 2 * 320a ⊕ 1 * 1242a

11*11b⊕3 * 11a ⊕ 10 * 1a ⊕ 6 * 44a ⊕ 2 * 44b ⊕ 4 * 120a ⊕ 1 * 220a ⊕ 4 * 220b ⊕ 1 * 252a ⊕ 1 * 1242a

7*11b⊕10 * 11a ⊕ 7 * 1a ⊕ 2 * 44a ⊕ 2 * 44b ⊕ 5 * 120a ⊕ 2 * 220a ⊕ 2 * 220b ⊕ 7 * 252a ⊕ 1 * 320b ⊕ 1 * 320a

6*11b⊕7 * 11a ⊕ 9 * 1a ⊕ 2 * 44a ⊕ 9 * 44b ⊕ 6 * 120a ⊕ 5 * 220a ⊕ 1 * 220b ⊕ 1 * 252a ⊕ 1 * 320b ⊕ 2 * 1242a ⊕ 1 * 1792a

7*11b⊕8 * 11a ⊕ 6 * 1a ⊕ 9 * 44a ⊕ 5 * 44b ⊕ 3 * 120a ⊕ 6 * 220b ⊕ 3 * 252a ⊕ 4 * 320a ⊕ 1 * 1242a

18*11b⊕6 * 11a ⊕ 6 * 1a ⊕ 5 * 44a ⊕ 4 * 44b ⊕ 6 * 120a ⊕ 3 * 220a ⊕ 5 * 252a ⊕ 2 * 320b ⊕ 1 * 1242a ⊕ 1 * 1792a

2*11b⊕14 * 11a ⊕ 9 * 1a ⊕ 4 * 44a ⊕ 7 * 44b ⊕ 7 * 120a ⊕ 2 * 220a ⊕ 1 * 220b ⊕ 3 * 252a ⊕ 3 * 1242a

6*11b⊕4 * 11a ⊕ 8 * 1a ⊕ 9 * 44a ⊕ 8 * 44b ⊕ 1 * 120a ⊕ 4 * 220a ⊕ 3 * 220b ⊕ 4 * 252a ⊕ 1 * 320b ⊕ 5 * 320a

16*11b⊕8 * 11a ⊕ 8 * 1a ⊕ 4 * 44a ⊕ 3 * 44b ⊕ 8 * 120a ⊕ 2 * 220a ⊕ 6 * 220b ⊕ 2 * 252a ⊕ 2 * 1792a

8*11b⊕12 * 11a ⊕ 8 * 1a ⊕ 3 * 44a ⊕ 10 * 44b ⊕ 2 * 120a ⊕ 5 * 220a ⊕ 1 * 220b ⊕ 6 * 252a ⊕ 3 * 320b

7*11b⊕8 * 11a ⊕ 11 * 1a ⊕ 7 * 44a ⊕ 5 * 44b ⊕ 5 * 120a ⊕ 3 * 220a ⊕ 3 * 220b ⊕ 1 * 252a ⊕ 3 * 1242a

5*11b⊕12 * 11a ⊕ 3 * 1a ⊕ 10 * 44a ⊕ 3 * 44b ⊕ 2 * 120a ⊕ 1 * 220a ⊕ 5 * 220b ⊕ 6 * 252a ⊕ 1 * 320b ⊕ 2 * 320a ⊕ 1 * 1242a

12*11b⊕4 * 11a ⊕ 9 * 1a ⊕ 2 * 44a ⊕ 8 * 44b ⊕ 12 * 120a ⊕ 7 * 220a ⊕ 1 * 220b ⊕ 2 * 252a ⊕ 4 * 320b ⊕ 2 * 1242a ⊕ 1 * 1792a

3*11b⊕10 * 11a ⊕ 11 * 1a ⊕ 4 * 44a ⊕ 8 * 44b ⊕ 2 * 120a ⊕ 5 * 220b ⊕ 4 * 252a ⊕ 3 * 320a ⊕ 2 * 1242a

15*11b⊕5 * 11a ⊕ 7 * 1a ⊕ 8 * 44a ⊕ 2 * 44b ⊕ 4 * 120a ⊕ 1 * 220a ⊕ 1 * 220b ⊕ 6 * 252a ⊕ 1 * 320b ⊕ 1 * 1792a

4*11b⊕12 * 11a ⊕ 5 * 1a ⊕ 5 * 44a ⊕ 4 * 44b ⊕ 10 * 120a ⊕ 1 * 220a ⊕ 2 * 220b ⊕ 3 * 252a ⊕ 2 * 1242a

4*11b⊕2 * 11a ⊕ 5 * 1a ⊕ 7 * 44a ⊕ 12 * 44b ⊕ 1 * 120a ⊕ 6 * 220a ⊕ 2 * 252a ⊕ 1 * 320b ⊕ 3 * 320a

13*11b⊕7 * 11a ⊕ 8 * 1a ⊕ 6 * 44a ⊕ 5 * 120a ⊕ 6 * 220b ⊕ 1 * 1242a ⊕ 1 * 1792a

7*11b⊕11 * 11a ⊕ 4 * 1a ⊕ 4 * 44a ⊕ 6 * 44b ⊕ 3 * 220a ⊕ 1 * 220b ⊕ 8 * 252a ⊕ 3 * 320b

5*11b⊕6 * 11a ⊕ 10 * 1a ⊕ 1 * 44a ⊕ 7 * 44b ⊕ 6 * 120a ⊕ 7 * 220a ⊕ 2 * 220b ⊕ 2 * 1242a

5*11b⊕8 * 11a ⊕ 3 * 1a ⊕ 9 * 44a ⊕ 3 * 44b ⊕ 6 * 220b ⊕ 3 * 252a ⊕ 2 * 320a

10*11b⊕4 * 11a ⊕ 7 * 1a ⊕ 3 * 44b ⊕ 6 * 120a ⊕ 3 * 220a ⊕ 2 * 252a ⊕ 3 * 320b ⊕ 1 * 1242a ⊕ 1 * 1792a

8*11a⊕5 * 1a ⊕ 3 * 44a ⊕ 3 * 44b ⊕ 2 * 120a ⊕ 2 * 220b ⊕ 3 * 252a ⊕ 1 * 1242a

5*11b⊕3 * 1a ⊕ 6 * 44a ⊕ 2 * 44b ⊕ 2 * 120a ⊕ 1 * 220a ⊕ 2 * 252a ⊕ 2 * 320a

4*11b⊕6 * 11a ⊕ 2 * 1a ⊕ 2 * 44a ⊕ 1 * 44b ⊕ 7 * 120a ⊕ 3 * 220b ⊕ 1 * 252a ⊕ 1 * 1242a ⊕ 1 * 1792a

3*11b⊕3 * 11a ⊕ 2 * 1a ⊕ 3 * 44a ⊕ 6 * 44b ⊕ 3 * 220a ⊕ 4 * 252a ⊕ 1 * 320b

3*11b⊕4 * 11a ⊕ 5 * 1a ⊕ 1 * 44a ⊕ 3 * 120a ⊕ 1 * 220b

1*11b⊕3 * 11a ⊕ 1 * 1a ⊕ 4 * 44a ⊕ 3 * 44b ⊕ 1 * 220a ⊕ 1 * 220b ⊕ 2 * 252a ⊕ 1 * 320b ⊕ 1 * 320a

5*11b⊕1 * 11a ⊕ 1 * 1a ⊕ 3 * 44b ⊕ 3 * 120a ⊕ 3 * 220a ⊕ 1 * 220b ⊕ 1 * 320b ⊕ 1 * 1242a

2*11b⊕4 * 11a ⊕ 3 * 1a ⊕ 2 * 44a ⊕ 1 * 44b ⊕ 2 * 220b ⊕ 2 * 252a

3*11b⊕1 * 11a ⊕ 3 * 1a ⊕ 1 * 44a ⊕ 2 * 252a ⊕ 1 * 320b

1*11b⊕4 * 11a ⊕ 1 * 1a ⊕ 2 * 44a ⊕ 3 * 120a ⊕ 1 * 220b ⊕ 1 * 1242a

1*11b⊕1 * 11a ⊕ 2 * 44a ⊕ 4 * 44b ⊕ 2 * 220a ⊕ 1 * 252a ⊕ 1 * 320b ⊕ 1 * 320a

4*11b⊕1 * 11a ⊕ 3 * 1a ⊕ 1 * 120a ⊕ 1 * 220b

1*11b⊕3 * 11a ⊕ 1 * 1a ⊕ 2 * 44a ⊕ 3 * 252a

1*11b⊕1 * 11a ⊕ 2 * 1a ⊕ 1 * 44b ⊕ 3 * 120a ⊕ 1 * 220a

1*11b⊕2 * 11a ⊕ 1 * 1a ⊕ 2 * 44a ⊕ 2 * 44b ⊕ 2 * 220b ⊕ 1 * 252a ⊕ 1 * 320a

5*11b⊕1 * 11a ⊕ 1 * 1a ⊕ 1 * 44b ⊕ 1 * 120a ⊕ 1 * 220a ⊕ 1 * 252a ⊕ 1 * 320b

4*11a⊕3 * 1a ⊕ 1 * 44a ⊕ 1 * 120a ⊕ 1 * 252a

1*1a⊕2 * 44a ⊕ 2 * 44b ⊕ 1 * 220a ⊕ 1 * 252a ⊕ 1 * 320a

3*11b⊕1 * 11a ⊕ 1 * 1a ⊕ 1 * 44a ⊕ 3 * 120a ⊕ 2 * 220b

2*11b⊕2 * 11a ⊕ 2 * 44b ⊕ 1 * 220a ⊕ 2 * 252a ⊕ 1 * 320b

1*11b⊕2 * 11a ⊕ 3 * 1a

2*11a⊕4 * 44a ⊕ 1 * 220b ⊕ 1 * 252a

2*11b⊕2 * 44b ⊕ 3 * 120a ⊕ 2 * 220a ⊕ 1 * 320b

2*11a⊕2 * 1a ⊕ 1 * 44b ⊕ 1 * 220b ⊕ 1 * 252a

2*11b⊕1 * 1a ⊕ 1 * 44a ⊕ 1 * 252a

2*11a⊕1 * 44a ⊕ 2 * 120a

1*44a⊕1 * 44b ⊕ 1 * 220a

1*120a⊕1 * 220b

1*11b⊕1 * 252a

1*11a⊕1 * 1a

1*44a

Socle length 65

FIGURE 5.5. Socle series of P(44b)

1 ∗ 44b
1*1a⊕1 ∗ 11b ⊕ 1 ∗ 44a
1*1a⊕1 ∗ 11a ⊕ 2 ∗ 120a ⊕ 1 ∗ 252a ⊕ 1 ∗ 1242a
2*1a⊕2 ∗ 11a ⊕ 3 ∗ 44b ⊕ 1 ∗ 120a ⊕ 1 ∗ 220a ⊕ 1 ∗ 252a ⊕ 1 ∗ 320a
2*1a⊕1 ∗ 11a ⊕ 4 ∗ 11b ⊕ 1 ∗ 44b ⊕ 5 ∗ 44a ⊕ 2 ∗ 220b ⊕ 1 ∗ 320a ⊕ 1 ∗ 1792a
5*1a⊕4 ∗ 11a ⊕ 5 ∗ 11b ⊕ 1 ∗ 44b ⊕ 2 ∗ 44a ⊕ 2 ∗ 120a ⊕ 1 ∗ 220a ⊕ 1 ∗ 220b ⊕ 1 ∗ 252a ⊕ 1 ∗ 320b
5*1a⊕5 ∗ 11a ⊕ 2 ∗ 11b ⊕ 5 ∗ 44b ⊕ 2 ∗ 120a ⊕ 2 ∗ 220a ⊕ 1 ∗ 320b ⊕ 1 ∗ 1242a
3*1a⊕2 ∗ 11a ⊕ 3 ∗ 11b ⊕ 2 ∗ 44b ⊕ 5 ∗ 44a ⊕ 1 ∗ 120a ⊕ 2 ∗ 220a ⊕ 1 ∗ 220b ⊕ 2 ∗ 252a ⊕ 2 ∗ 320a ⊕ 1 ∗ 1242a
4*1a⊕3 ∗ 11a ⊕ 8 ∗ 11b ⊕ 2 ∗ 44a ⊕ 5 ∗ 120a ⊕ 2 ∗ 220b ⊕ 3 ∗ 252a ⊕ 1 ∗ 320a ⊕ 1 ∗ 1792a
6*1a⊕7 ∗ 11a ⊕ 3 ∗ 11b ⊕ 5 ∗ 44b ⊕ 4 ∗ 120a ⊕ 3 ∗ 252a ⊕ 1 ∗ 320b ⊕ 1 ∗ 1242a ⊕ 1 ∗ 1792a
3*1a⊕5 ∗ 11a ⊕ 3 ∗ 11b ⊕ 6 ∗ 44b ⊕ 6 ∗ 44a ⊕ 2 ∗ 120a ⊕ 2 ∗ 220a ⊕ 1 ∗ 220b ⊕ 2 ∗ 252a ⊕ 1 ∗ 320a ⊕ 2 ∗ 1242a
6*1a⊕6 ∗ 11a ⊕ 6 ∗ 11b ⊕ 2 ∗ 44b ⊕ 7 ∗ 44a ⊕ 2 ∗ 120a ⊕ 4 ∗ 220b ⊕ 3 ∗ 320a
4*1a⊕5 ∗ 11a ⊕ 10 ∗ 11b ⊕ 7 ∗ 44b ⊕ 2 ∗ 44a ⊕ 3 ∗ 120a ⊕ 5 ∗ 220a ⊕ 1 ∗ 220b ⊕ 2 ∗ 252a ⊕ 2 ∗ 320b ⊕ 2 ∗ 1792a
10*1a⊕7 ∗ 11a ⊕ 7 ∗ 11b ⊕ 6 ∗ 44b ⊕ 3 ∗ 44a ⊕ 2 ∗ 120a ⊕ 3 ∗ 220a ⊕ 2 ∗ 220b ⊕ 1 ∗ 252a ⊕ 3 ∗ 320b ⊕ 3 ∗ 1242a
7*1a⊕6 ∗ 11a ⊕ 7 ∗ 11b ⊕ 2 ∗ 44b ⊕ 8 ∗ 44a ⊕ 3 ∗ 120a ⊕ 1 ∗ 220a ⊕ 2 ∗ 220b ⊕ 5 ∗ 252a ⊕ 2 ∗ 320a
9*1a⊕9 ∗ 11a ⊕ 9 ∗ 11b ⊕ 2 ∗ 44b ⊕ 3 ∗ 44a ⊕ 9 ∗ 120a ⊕ 2 ∗ 220a ⊕ 2 ∗ 220b ⊕ 3 ∗ 252a ⊕ 1 ∗ 320b ⊕ 2 ∗ 1242a ⊕ 1 ∗ 1792a
8*1a⊕8 ∗ 11a ⊕ 4 ∗ 11b ⊕ 9 ∗ 44b ⊕ 4 ∗ 44a ⊕ 5 ∗ 120a ⊕ 3 ∗ 220a ⊕ 3 ∗ 220b ⊕ 3 ∗ 252a ⊕ 1 ∗ 320b ⊕ 1 ∗ 320a ⊕ 1 ∗ 1242a
8*1a⊕6 ∗ 11a ⊕ 9 ∗ 11b ⊕ 5 ∗ 44b ⊕ 7 ∗ 44a ⊕ 4 ∗ 120a ⊕ 1 ∗ 220a ⊕ 3 ∗ 220b ⊕ 3 ∗ 252a ⊕ 2 ∗ 320a ⊕ 1 ∗ 1242a
4*1a⊕11 ∗ 11a ⊕ 8 ∗ 11b ⊕ 4 ∗ 44b ⊕ 5 ∗ 44a ⊕ 4 ∗ 120a ⊕ 1 ∗ 220a ⊕ 3 ∗ 220b ⊕ 4 ∗ 252a ⊕ 2 ∗ 320a ⊕ 1 ∗ 1242a ⊕ 1 ∗ 1792a
8*1a⊕5 ∗ 11a ⊕ 11 ∗ 11b ⊕ 9 ∗ 44b ⊕ 6 ∗ 44a ⊕ 4 ∗ 120a ⊕ 7 ∗ 220a ⊕ 2 ∗ 252a ⊕ 2 ∗ 320b ⊕ 2 ∗ 1242a ⊕ 1 ∗ 1792a
11*1a⊕8 ∗ 11a ⊕ 6 ∗ 11b ⊕ 4 ∗ 44b ⊕ 5 ∗ 44a ⊕ 3 ∗ 120a ⊕ 1 ∗ 220a ⊕ 5 ∗ 220b ⊕ 2 ∗ 252a ⊕ 1 ∗ 320a ⊕ 1 ∗ 1242a
6*1a⊕8 ∗ 11a ⊕ 10 ∗ 11b ⊕ 3 ∗ 44b ⊕ 7 ∗ 44a ⊕ 3 ∗ 120a ⊕ 2 ∗ 220a ⊕ 2 ∗ 220b ⊕ 7 ∗ 252a ⊕ 1 ∗ 320b
8*1a⊕11 ∗ 11a ⊕ 5 ∗ 11b ⊕ 7 ∗ 44b ⊕ 2 ∗ 44a ⊕ 9 ∗ 120a ⊕ 4 ∗ 220a ⊕ 4 ∗ 220b ⊕ 2 ∗ 252a ⊕ 1 ∗ 320b ⊕ 3 ∗ 1242a
7*1a⊕6 ∗ 11a ⊕ 7 ∗ 11b ⊕ 10 ∗ 44b ⊕ 9 ∗ 44a ⊕ 2 ∗ 120a ⊕ 4 ∗ 220a ⊕ 4 ∗ 220b ⊕ 4 ∗ 252a ⊕ 2 ∗ 320b ⊕ 4 ∗ 320a
12*1a⊕8 ∗ 11a ⊕ 17 ∗ 11b ⊕ 2 ∗ 44b ⊕ 5 ∗ 44a ⊕ 5 ∗ 120a ⊕ 1 ∗ 220a ⊕ 3 ∗ 220b ⊕ 2 ∗ 252a ⊕ 1 ∗ 320b ⊕ 1 ∗ 1792a
4*1a⊕15 ∗ 11a ⊕ 5 ∗ 11b ⊕ 4 ∗ 44b ⊕ 5 ∗ 44a ⊕ 5 ∗ 120a ⊕ 2 ∗ 220a ⊕ 6 ∗ 252a ⊕ 1 ∗ 320b ⊕ 2 ∗ 1242a
5*1a⊕3 ∗ 11a ⊕ 4 ∗ 11b ⊕ 8 ∗ 44b ⊕ 6 ∗ 44a ⊕ 7 ∗ 120a ⊕ 5 ∗ 220a ⊕ 2 ∗ 220b ⊕ 1 ∗ 252a ⊕ 1 ∗ 320a ⊕ 2 ∗ 1242a
7*1a⊕7 ∗ 11a ⊕ 6 ∗ 11b ⊕ 5 ∗ 44b ⊕ 6 ∗ 44a ⊕ 2 ∗ 120a ⊕ 7 ∗ 220b ⊕ 3 ∗ 252a ⊕ 3 ∗ 320a ⊕ 1 ∗ 1792a
7*1a⊕6 ∗ 11a ⊕ 15 ∗ 11b ⊕ 7 ∗ 44b ⊕ 2 ∗ 44a ⊕ 5 ∗ 120a ⊕ 5 ∗ 220a ⊕ 1 ∗ 220b ⊕ 3 ∗ 252a ⊕ 3 ∗ 320b ⊕ 1 ∗ 1792a
9*1a⊕11 ∗ 11a ⊕ 4 ∗ 11b ⊕ 6 ∗ 44b ⊕ 4 ∗ 44a ⊕ 3 ∗ 120a ⊕ 2 ∗ 220a ⊕ 3 ∗ 220b ⊕ 3 ∗ 252a ⊕ 1 ∗ 1242a
6*1a⊕5 ∗ 11a ⊕ 7 ∗ 11b ⊕ 4 ∗ 44b ⊕ 9 ∗ 44a ⊕ 1 ∗ 120a ⊕ 2 ∗ 220a ⊕ 5 ∗ 252a ⊕ 1 ∗ 320b ⊕ 1 ∗ 320a
2*1a⊕8 ∗ 11a ⊕ 6 ∗ 11b ⊕ 1 ∗ 44b ⊕ 4 ∗ 44a ⊕ 12 ∗ 120a ⊕ 1 ∗ 220a ⊕ 3 ∗ 220b ⊕ 2 ∗ 252a ⊕ 1 ∗ 320b ⊕ 2 ∗ 1242a
5*1a⊕4 ∗ 11a ⊕ 4 ∗ 11b ⊕ 9 ∗ 44b ⊕ 3 ∗ 44a ⊕ 1 ∗ 120a ⊕ 4 ∗ 220a ⊕ 1 ∗ 220b ⊕ 5 ∗ 252a ⊕ 2 ∗ 320b ⊕ 2 ∗ 320a
10*1a⊕4 ∗ 11a ⊕ 9 ∗ 11b ⊕ 1 ∗ 44a ⊕ 4 ∗ 120a ⊕ 1 ∗ 220a ⊕ 3 ∗ 220b ⊕ 1 ∗ 1792a
1*1a⊕8 ∗ 11a ⊕ 3 ∗ 11b ⊕ 3 ∗ 44b ⊕ 6 ∗ 44a ⊕ 3 ∗ 220b ⊕ 5 ∗ 252a ⊕ 1 ∗ 320b ⊕ 1 ∗ 320a ⊕ 1 ∗ 1242a
3*1a⊕3 ∗ 11a ⊕ 8 ∗ 11b ⊕ 6 ∗ 44b ⊕ 2 ∗ 44a ⊕ 5 ∗ 120a ⊕ 6 ∗ 220a ⊕ 1 ∗ 320b ⊕ 2 ∗ 1242a
5*1a⊕9 ∗ 11a ⊕ 2 ∗ 11b ⊕ 2 ∗ 44b ⊕ 6 ∗ 44a ⊕ 1 ∗ 120a ⊕ 3 ∗ 220b ⊕ 2 ∗ 252a ⊕ 1 ∗ 320a
6*1a⊕2 ∗ 11a ⊕ 4 ∗ 11b ⊕ 3 ∗ 44b ⊕ 1 ∗ 44a ⊕ 3 ∗ 120a ⊕ 2 ∗ 220a ⊕ 3 ∗ 252a ⊕ 2 ∗ 320b ⊕ 1 ∗ 1792a
3*1a⊕5 ∗ 11a ⊕ 1 ∗ 11b ⊕ 1 ∗ 44b ⊕ 3 ∗ 44a ⊕ 3 ∗ 120a ⊕ 3 ∗ 220b ⊕ 1 ∗ 252a ⊕ 1 ∗ 1242a
1*1a⊕1 ∗ 11a ⊕ 4 ∗ 11b ⊕ 4 ∗ 44b ⊕ 3 ∗ 44a ⊕ 2 ∗ 220a ⊕ 2 ∗ 252a ⊕ 1 ∗ 320b ⊕ 1 ∗ 320a
4*1a⊕5 ∗ 11a ⊕ 4 ∗ 11b ⊕ 1 ∗ 44a ⊕ 3 ∗ 120a ⊕ 1 ∗ 220b ⊕ 1 ∗ 1242a
1*1a⊕3 ∗ 11a ⊕ 1 ∗ 11b ⊕ 2 ∗ 44b ⊕ 4 ∗ 44a ⊕ 1 ∗ 220a ⊕ 3 ∗ 252a ⊕ 1 ∗ 320b
4*1a⊕1 ∗ 11a ⊕ 2 ∗ 11b ⊕ 1 ∗ 44b ⊕ 5 ∗ 120a ⊕ 1 ∗ 220a ⊕ 1 ∗ 220b
1*1a⊕3 ∗ 11a ⊕ 2 ∗ 11b ⊕ 2 ∗ 44b ⊕ 2 ∗ 44a ⊕ 2 ∗ 220b ⊕ 2 ∗ 252a ⊕ 1 ∗ 320a
2*1a⊕2 ∗ 11a ⊕ 5 ∗ 11b ⊕ 2 ∗ 44b ⊕ 1 ∗ 120a ⊕ 2 ∗ 220a ⊕ 1 ∗ 252a ⊕ 1 ∗ 320b ⊕ 1 ∗ 1242a
3*1a⊕4 ∗ 11a ⊕ 3 ∗ 44a ⊕ 1 ∗ 120a ⊕ 1 ∗ 220b ⊕ 1 ∗ 252a
1*1a⊕1 ∗ 11b ⊕ 2 ∗ 44b ⊕ 2 ∗ 44a ⊕ 1 ∗ 220a ⊕ 1 ∗ 252a ⊕ 1 ∗ 320a
1*1a⊕2 ∗ 11a ⊕ 4 ∗ 11b ⊕ 1 ∗ 44a ⊕ 3 ∗ 120a ⊕ 2 ∗ 220b
1*1a⊕3 ∗ 11a ⊕ 2 ∗ 11b ⊕ 2 ∗ 44b ⊕ 1 ∗ 44a ⊕ 1 ∗ 220a ⊕ 3 ∗ 252a ⊕ 1 ∗ 320b
3*1a⊕2 ∗ 11a ⊕ 1 ∗ 11b ⊕ 1 ∗ 120a
2*11a⊕1 ∗ 44b ⊕ 4 ∗ 44a ⊕ 1 ∗ 220b ⊕ 1 ∗ 252a ⊕ 1 ∗ 320a
3*11b⊕2 ∗ 44b ⊕ 3 ∗ 120a ⊕ 2 ∗ 220a ⊕ 1 ∗ 320b
2*1a⊕2 ∗ 11a ⊕ 1 ∗ 11b ⊕ 1 ∗ 44b ⊕ 1 ∗ 220b ⊕ 2 ∗ 252a
2*1a⊕1 ∗ 11a ⊕ 2 ∗ 11b ⊕ 1 ∗ 44a ⊕ 2 ∗ 252a
3*11a⊕1 ∗ 11b ⊕ 2 ∗ 44a ⊕ 2 ∗ 120a ⊕ 1 ∗ 220b
1*11b⊕4 ∗ 44b ⊕ 1 ∗ 44a ⊕ 2 ∗ 220a ⊕ 1 ∗ 320a
2*1a⊕3 ∗ 11b ⊕ 1 ∗ 120a ⊕ 1 ∗ 220b
2*11a⊕1 ∗ 11b ⊕ 2 ∗ 252a
1*1a⊕1 ∗ 11a ⊕ 1 ∗ 44b ⊕ 1 ∗ 120a ⊕ 1 ∗ 220a
1*1a⊕1 ∗ 11a ⊕ 1 ∗ 11b ⊕ 1 ∗ 44b ⊕ 2 ∗ 44a ⊕ 1 ∗ 220b
2*11b⊕1 ∗ 44b ⊕ 1 ∗ 120a ⊕ 1 ∗ 220a
1*1a⊕1 ∗ 252a
1*1a⊕1 ∗ 252a
1*11a⊕1 ∗ 120a ⊕ 1 ∗ 220b
1*44b

Socle length 65

FIGURE 5.6. Socle series of P(120a)

$1 * 120a$

$1{*}11\mathrm{a} \oplus 1 * 44b \oplus 1 * 252a \oplus 1 * 320a$

$2{*}11\mathrm{b} \oplus 1 * 1a \oplus 2 * 44a \oplus 1 * 1792a$

$2{*}11\mathrm{a} \oplus 1 * 11b \oplus 3 * 1a \oplus 1 * 44a \oplus 1 * 120a \oplus 1 * 220b \oplus 1 * 320b$

$1{*}11\mathrm{a} \oplus 2 * 11b \oplus 1 * 1a \oplus 3 * 44b \oplus 1 * 120a \oplus 2 * 220a \oplus 1 * 252a \oplus 1 * 320b \oplus 1 * 1242a$

$4{*}11\mathrm{a} \oplus 2 * 11b \oplus 4 * 1a \oplus 2 * 44a \oplus 2 * 120a \oplus 1 * 220b \oplus 1 * 1242a$

$3{*}11\mathrm{a} \oplus 2 * 11b \oplus 1 * 1a \oplus 2 * 44a \oplus 2 * 44b \oplus 2 * 120a \oplus 1 * 220a \oplus 3 * 252a \oplus 2 * 320a$

$3{*}11\mathrm{a} \oplus 4 * 11b \oplus 5 * 1a \oplus 1 * 44a \oplus 3 * 44b \oplus 3 * 120a \oplus 1 * 220b \oplus 1 * 252a \oplus 1 * 1242a \oplus 1 * 1792a$

$3{*}11\mathrm{a} \oplus 3 * 11b \oplus 3 * 1a \oplus 5 * 44a \oplus 3 * 44b \oplus 1 * 220a \oplus 1 * 220b \oplus 1 * 252a \oplus 2 * 320a$

$3{*}11\mathrm{a} \oplus 7 * 11b \oplus 4 * 1a \oplus 4 * 44a \oplus 2 * 44b \oplus 2 * 120a \oplus 1 * 220a \oplus 2 * 220b \oplus 1 * 252a \oplus 1 * 320b \oplus 1 * 1792a$

$7{*}11\mathrm{a} \oplus 4 * 11b \oplus 5 * 1a \oplus 2 * 44a \oplus 2 * 44b \oplus 1 * 120a \oplus 2 * 220a \oplus 2 * 220b \oplus 1 * 252a \oplus 2 * 320b$

$4{*}11\mathrm{a} \oplus 4 * 11b \oplus 4 * 1a \oplus 2 * 44a \oplus 5 * 44b \oplus 3 * 120a \oplus 4 * 220a \oplus 1 * 220b \oplus 2 * 252a \oplus 1 * 320b \oplus 2 * 1242a$

$5{*}11\mathrm{a} \oplus 4 * 11b \oplus 7 * 1a \oplus 3 * 44a \oplus 1 * 44b \oplus 4 * 120a \oplus 4 * 220b \oplus 1 * 252a \oplus 2 * 1242a$

$4{*}11\mathrm{a} \oplus 7 * 11b \oplus 4 * 1a \oplus 2 * 44a \oplus 4 * 44b \oplus 5 * 120a \oplus 2 * 220a \oplus 5 * 252a \oplus 3 * 320a \oplus 1 * 320b$

$7{*}11\mathrm{a} \oplus 7 * 11b \oplus 8 * 1a \oplus 3 * 44a \oplus 5 * 44b \oplus 7 * 120a \oplus 1 * 220b \oplus 1 * 252a \oplus 2 * 1242a \oplus 2 * 1792a$

$6{*}11\mathrm{a} \oplus 3 * 11b \oplus 2 * 1a \oplus 9 * 44a \oplus 6 * 44b \oplus 1 * 220b \oplus 5 * 252a \oplus 4 * 320a$

$4{*}11\mathrm{a} \oplus 14 * 11b \oplus 7 * 1a \oplus 5 * 44a \oplus 3 * 44b \oplus 6 * 120a \oplus 3 * 220a \oplus 1 * 1242a \oplus 1 * 1792a$

$13{*}11\mathrm{a} \oplus 1 * 11b \oplus 7 * 1a \oplus 4 * 44a \oplus 6 * 44b \oplus 2 * 220a \oplus 3 * 220b \oplus 4 * 252a \oplus 1 * 320a \oplus 1 * 320b$

$1{*}11\mathrm{a} \oplus 10 * 11b \oplus 9 * 1a \oplus 4 * 44a \oplus 7 * 44b \oplus 2 * 120a \oplus 7 * 220a \oplus 2 * 220b \oplus 2 * 252a \oplus 2 * 320b$

$12{*}11\mathrm{a} \oplus 5 * 11b \oplus 6 * 1a \oplus 4 * 44a \oplus 1 * 44b \oplus 3 * 120a \oplus 9 * 220b \oplus 2 * 252a \oplus 2 * 1242a$

$6{*}11\mathrm{a} \oplus 11 * 11b \oplus 6 * 1a \oplus 3 * 44a \oplus 8 * 44b \oplus 2 * 120a \oplus 7 * 220a \oplus 3 * 252a \oplus 1 * 320a \oplus 4 * 320b$

$11{*}11\mathrm{a} \oplus 4 * 11b \oplus 11 * 1a \oplus 3 * 44a \oplus 2 * 44b \oplus 6 * 120a \oplus 4 * 220b \oplus 4 * 1242a \oplus 1 * 1792a$

$3{*}11\mathrm{a} \oplus 5 * 11b \oplus 1 * 1a \oplus 9 * 44a \oplus 5 * 44b \oplus 3 * 120a \oplus 1 * 220a \oplus 8 * 252a \oplus 3 * 320a \oplus 1 * 320b$

$4{*}11\mathrm{a} \oplus 11 * 11b \oplus 10 * 1a \oplus 2 * 44a \oplus 2 * 44b \oplus 17 * 120a \oplus 1 * 220b \oplus 1 * 252a \oplus 2 * 1242a \oplus 1 * 1792a$

$8{*}11\mathrm{a} \oplus 2 * 11b \oplus 2 * 1a \oplus 5 * 44a \oplus 12 * 44b \oplus 8 * 252a \oplus 4 * 320a$

$2{*}11\mathrm{a} \oplus 18 * 11b \oplus 10 * 1a \oplus 5 * 44a \oplus 2 * 44b \oplus 5 * 120a \oplus 3 * 220a \oplus 1 * 220b \oplus 1 * 320b \oplus 1 * 1792a$

$15{*}11\mathrm{a} \oplus 1 * 11b \oplus 2 * 1a \oplus 7 * 44a \oplus 4 * 44b \oplus 1 * 120a \oplus 1 * 220a \oplus 4 * 220b \oplus 4 * 252a$

$2{*}11\mathrm{a} \oplus 7 * 11b \oplus 8 * 1a \oplus 2 * 44a \oplus 10 * 44b \oplus 1 * 120a \oplus 11 * 220a \oplus 2 * 320b$

$11{*}11\mathrm{a} \oplus 4 * 11b \oplus 7 * 1a \oplus 5 * 44a \oplus 1 * 44b \oplus 12 * 220b \oplus 1 * 252a \oplus 2 * 1242a$

$4{*}11\mathrm{a} \oplus 13 * 11b \oplus 5 * 1a \oplus 3 * 44a \oplus 5 * 44b \oplus 5 * 220a \oplus 5 * 252a \oplus 4 * 320b$

$13{*}11\mathrm{a} \oplus 2 * 11b \oplus 10 * 1a \oplus 1 * 44a \oplus 9 * 120a \oplus 3 * 220b \oplus 3 * 1242a$

$3{*}11\mathrm{a} \oplus 3 * 11b \oplus 1 * 1a \oplus 12 * 44a \oplus 6 * 44b \oplus 6 * 252a \oplus 3 * 320a \oplus 1 * 320b$

$3{*}11\mathrm{a} \oplus 8 * 11b \oplus 4 * 1a \oplus 1 * 44a \oplus 15 * 120a \oplus 2 * 1242a \oplus 1 * 1792a$

$5{*}11\mathrm{a} \oplus 2 * 1a \oplus 4 * 44a \oplus 6 * 44b \oplus 6 * 252a \oplus 2 * 320a$

$7{*}11\mathrm{b} \oplus 7 * 1a \oplus 2 * 44b \oplus 3 * 120a \oplus 2 * 220a \oplus 1 * 1792a$

$8{*}11\mathrm{a} \oplus 2 * 11b \oplus 1 * 1a \oplus 5 * 44a \oplus 2 * 44b \oplus 5 * 220b \oplus 3 * 252a$

$1{*}11\mathrm{a} \oplus 7 * 11b \oplus 4 * 1a \oplus 1 * 44a \oplus 7 * 44b \oplus 7 * 220a \oplus 1 * 252a \oplus 3 * 320b$

$7{*}11\mathrm{a} \oplus 2 * 11b \oplus 4 * 1a \oplus 3 * 44a \oplus 2 * 120a \oplus 3 * 220b \oplus 1 * 1242a$

$1{*}11\mathrm{a} \oplus 3 * 11b \oplus 2 * 1a \oplus 3 * 44a \oplus 3 * 44b \oplus 2 * 220a \oplus 3 * 252a \oplus 2 * 320a \oplus 2 * 320b$

$3{*}11\mathrm{a} \oplus 4 * 11b \oplus 3 * 1a \oplus 6 * 120a \oplus 2 * 220b \oplus 1 * 1242a \oplus 1 * 1792a$

$3{*}11\mathrm{a} \oplus 2 * 11b \oplus 1 * 1a \oplus 3 * 44a \oplus 3 * 44b \oplus 4 * 252a \oplus 1 * 320a$

$2{*}11\mathrm{a} \oplus 4 * 11b \oplus 3 * 1a \oplus 2 * 120a \oplus 1 * 1242a$

$4{*}11\mathrm{a} \oplus 1 * 1a \oplus 5 * 44a \oplus 1 * 220b \oplus 1 * 252a$

$1{*}11\mathrm{b} \oplus 2 * 1a \oplus 3 * 44b \oplus 2 * 120a \oplus 3 * 220a \oplus 1 * 320b$

$2{*}11\mathrm{a} \oplus 2 * 11b \oplus 3 * 1a \oplus 1 * 44a \oplus 3 * 220b$

$2{*}11\mathrm{a} \oplus 3 * 11b \oplus 1 * 44a \oplus 1 * 44b \oplus 1 * 220a \oplus 3 * 252a \oplus 1 * 320b$

$3{*}11\mathrm{a} \oplus 1 * 11b \oplus 3 * 1a \oplus 4 * 120a$

$1{*}11\mathrm{a} \oplus 3 * 44a \oplus 3 * 44b \oplus 2 * 252a \oplus 2 * 320a$

$6{*}11\mathrm{b} \oplus 1 * 1a \oplus 4 * 120a$

$4{*}11\mathrm{a} \oplus 2 * 1a \oplus 1 * 44a \oplus 1 * 44b \oplus 3 * 252a$

$1{*}11\mathrm{b} \oplus 3 * 1a \oplus 1 * 44b \oplus 1 * 220a$

$3{*}11\mathrm{a} \oplus 1 * 1a \oplus 3 * 44a \oplus 3 * 220b$

$3{*}11\mathrm{b} \oplus 3 * 44b \oplus 3 * 220a \oplus 1 * 320b$

$3{*}11\mathrm{a} \oplus 1 * 11b \oplus 3 * 1a \oplus 1 * 220b$

$1{*}11\mathrm{a} \oplus 1 * 11b \oplus 2 * 44a \oplus 3 * 252a$

$1{*}11\mathrm{a} \oplus 1 * 11b \oplus 5 * 120a$

$1{*}11\mathrm{a} \oplus 1 * 44a \oplus 3 * 44b \oplus 2 * 252a \oplus 1 * 320a$

$3{*}11\mathrm{b} \oplus 2 * 1a \oplus 1 * 120a$

$3{*}11\mathrm{a} \oplus 1 * 44a \oplus 1 * 252a$

$1{*}1\mathrm{a} \oplus 2 * 44b \oplus 1 * 220a \oplus 1 * 320b$

$1{*}11\mathrm{a} \oplus 1 * 11b \oplus 1 * 44a \oplus 2 * 220b$

$2{*}11\mathrm{b} \oplus 1 * 44b \oplus 1 * 220a$

$1{*}11\mathrm{a} \oplus 1 * 1a$

$1{*}44\mathrm{a} \oplus 1 * 252a$

$1{*}120\mathrm{a}$

Socle length 65

FIGURE 5.7. Socle series of P(220a)

1*220a
1*11a⊕1 ∗ 1a ⊕ 1 ∗ 220b
1*11b⊕1 ∗ 44b ⊕ 1 ∗ 252a
1*11a⊕1 ∗ 1a ⊕ 1 ∗ 120a ⊕ 1 ∗ 1242a
2*44a⊕1 ∗ 44b ⊕ 1 ∗ 220a ⊕ 2 ∗ 320a
4*11b⊕3 ∗ 1a ⊕ 1 ∗ 44a ⊕ 1 ∗ 220b ⊕ 1 ∗ 120a ⊕ 1 ∗ 1792a
1*11b⊕3 ∗ 11a ⊕ 2 ∗ 1a ⊕ 1 ∗ 44b ⊕ 1 ∗ 252a ⊕ 1 ∗ 320b
1*11b⊕1 ∗ 11a ⊕ 1 ∗ 1a ⊕ 1 ∗ 44a ⊕ 2 ∗ 44b ⊕ 1 ∗ 220a ⊕ 1 ∗ 120a ⊕ 1 ∗ 252a ⊕ 1 ∗ 1242a
1*11b⊕3 ∗ 11a ⊕ 2 ∗ 1a ⊕ 2 ∗ 44a ⊕ 1 ∗ 220a ⊕ 2 ∗ 220b ⊕ 1 ∗ 120a ⊕ 1 ∗ 320a
4*11b⊕1 ∗ 11a ⊕ 2 ∗ 1a ⊕ 3 ∗ 44b ⊕ 2 ∗ 220a ⊕ 2 ∗ 120a ⊕ 1 ∗ 252a ⊕ 1 ∗ 320b ⊕ 1 ∗ 1792a
2*11b⊕4 ∗ 11a ⊕ 4 ∗ 1a ⊕ 1 ∗ 44a ⊕ 2 ∗ 44b ⊕ 1 ∗ 220b ⊕ 2 ∗ 120a ⊕ 1 ∗ 320b ⊕ 1 ∗ 1242a
2*11b⊕2 ∗ 11a ⊕ 1 ∗ 1a ⊕ 4 ∗ 44a ⊕ 1 ∗ 44b ⊕ 1 ∗ 120a ⊕ 3 ∗ 252a ⊕ 2 ∗ 320a
5*11b⊕3 ∗ 11a ⊕ 4 ∗ 1a ⊕ 1 ∗ 44a ⊕ 1 ∗ 44b ⊕ 1 ∗ 220a ⊕ 4 ∗ 120a ⊕ 1 ∗ 1242a ⊕ 1 ∗ 1792a
4*11a⊕3 ∗ 1a ⊕ 2 ∗ 44a ⊕ 4 ∗ 44b ⊕ 1 ∗ 220a ⊕ 2 ∗ 220b ⊕ 2 ∗ 252a ⊕ 1 ∗ 320a
6*11b⊕4 ∗ 1a ⊕ 2 ∗ 44a ⊕ 2 ∗ 44b ⊕ 1 ∗ 220a ⊕ 1 ∗ 220b ⊕ 1 ∗ 120a ⊕ 1 ∗ 252a
2*11b⊕7 ∗ 11a ⊕ 1 ∗ 1a ⊕ 2 ∗ 44a ⊕ 1 ∗ 44b ⊕ 3 ∗ 220b ⊕ 1 ∗ 120a ⊕ 1 ∗ 252a ⊕ 1 ∗ 1242a
6*11b⊕1 ∗ 11a ⊕ 3 ∗ 1a ⊕ 3 ∗ 44a ⊕ 6 ∗ 44b ⊕ 5 ∗ 220a ⊕ 2 ∗ 320b ⊕ 1 ∗ 1242a
4*11b⊕5 ∗ 11a ⊕ 7 ∗ 1a ⊕ 3 ∗ 44a ⊕ 3 ∗ 220b ⊕ 3 ∗ 120a ⊕ 1 ∗ 1242a
3*11b⊕5 ∗ 11a ⊕ 3 ∗ 1a ⊕ 3 ∗ 44a ⊕ 1 ∗ 44b ⊕ 1 ∗ 220a ⊕ 2 ∗ 120a ⊕ 6 ∗ 252a ⊕ 1 ∗ 320b ⊕ 1 ∗ 320a
2*11b⊕3 ∗ 11a ⊕ 6 ∗ 1a ⊕ 3 ∗ 44b ⊕ 1 ∗ 220a ⊕ 1 ∗ 220b ⊕ 9 ∗ 120a ⊕ 2 ∗ 1242a ⊕ 1 ∗ 1792a
2*11b⊕3 ∗ 11a ⊕ 1 ∗ 1a ⊕ 5 ∗ 44a ⊕ 6 ∗ 44b ⊕ 2 ∗ 220b ⊕ 4 ∗ 252a ⊕ 4 ∗ 320a
13*11b⊕2 ∗ 11a ⊕ 5 ∗ 1a ⊕ 2 ∗ 44a ⊕ 1 ∗ 44b ⊕ 1 ∗ 220a ⊕ 4 ∗ 120a ⊕ 1 ∗ 1792a
10*11a⊕3 ∗ 1a ⊕ 4 ∗ 44a ⊕ 3 ∗ 44b ⊕ 1 ∗ 220a ⊕ 3 ∗ 252a ⊕ 1 ∗ 1242a
3*11b⊕3 ∗ 1a ⊕ 4 ∗ 44a ⊕ 5 ∗ 44b ⊕ 5 ∗ 220a ⊕ 1 ∗ 220b ⊕ 1 ∗ 120a ⊕ 1 ∗ 320b
4*11b⊕6 ∗ 11a ⊕ 5 ∗ 1a ⊕ 3 ∗ 44a ⊕ 1 ∗ 44b ⊕ 7 ∗ 220b ⊕ 1 ∗ 252a
8*11b⊕3 ∗ 11a ⊕ 4 ∗ 1a ⊕ 5 ∗ 44b ⊕ 5 ∗ 220a ⊕ 1 ∗ 120a ⊕ 2 ∗ 252a ⊕ 3 ∗ 320b
1*11b⊕8 ∗ 11a ⊕ 7 ∗ 1a ⊕ 2 ∗ 44a ⊕ 1 ∗ 44b ⊕ 3 ∗ 220b ⊕ 4 ∗ 120a ⊕ 2 ∗ 1242a
3*11b⊕2 ∗ 11a ⊕ 1 ∗ 1a ⊕ 7 ∗ 44a ⊕ 2 ∗ 44b ⊕ 5 ∗ 252a ⊕ 1 ∗ 320b ⊕ 2 ∗ 320a
6*11b⊕3 ∗ 11a ⊕ 2 ∗ 1a ⊕ 1 ∗ 44a ⊕ 12 ∗ 120a ⊕ 2 ∗ 1242a ⊕ 1 ∗ 1792a
3*11a⊕1 ∗ 1a ⊕ 3 ∗ 44a ⊕ 6 ∗ 44b ⊕ 4 ∗ 252a ⊕ 2 ∗ 320a
8*11b⊕6 ∗ 1a ⊕ 1 ∗ 44b ⊕ 1 ∗ 220a ⊕ 3 ∗ 120a ⊕ 1 ∗ 1792a
7*11a⊕3 ∗ 44a ⊕ 2 ∗ 44b ⊕ 3 ∗ 220b ⊕ 3 ∗ 252a
5*11b⊕4 ∗ 1a ⊕ 1 ∗ 44a ⊕ 6 ∗ 44b ⊕ 7 ∗ 220a ⊕ 1 ∗ 320b
3*11b⊕8 ∗ 11a ⊕ 3 ∗ 1a ⊕ 3 ∗ 44a ⊕ 4 ∗ 220b
3*11b⊕2 ∗ 11a ⊕ 3 ∗ 1a ⊕ 3 ∗ 44a ⊕ 2 ∗ 44b ⊕ 2 ∗ 220a ⊕ 3 ∗ 252a ⊕ 3 ∗ 320b
2*11b⊕3 ∗ 11a ⊕ 3 ∗ 1a ⊕ 2 ∗ 220b ⊕ 5 ∗ 120a ⊕ 2 ∗ 1242a
2*11b⊕1 ∗ 11a ⊕ 3 ∗ 44a ⊕ 3 ∗ 44b ⊕ 2 ∗ 252a ⊕ 2 ∗ 320a
3*11b⊕2 ∗ 11a ⊕ 3 ∗ 1a ⊕ 3 ∗ 120a ⊕ 1 ∗ 1242a ⊕ 1 ∗ 1792a
2*11a⊕1 ∗ 1a ⊕ 3 ∗ 44a ⊕ 1 ∗ 44b ⊕ 1 ∗ 252a
1*11b⊕2 ∗ 1a ⊕ 1 ∗ 44b ⊕ 1 ∗ 220a ⊕ 2 ∗ 120a
1*11b⊕2 ∗ 11a ⊕ 1 ∗ 1a ⊕ 1 ∗ 44a ⊕ 1 ∗ 44b ⊕ 2 ∗ 220b ⊕ 1 ∗ 252a
3*11b⊕1 ∗ 11a ⊕ 1 ∗ 1a ⊕ 2 ∗ 44b ⊕ 2 ∗ 220a ⊕ 1 ∗ 252a ⊕ 1 ∗ 320b
4*11a⊕2 ∗ 1a ⊕ 1 ∗ 44a ⊕ 1 ∗ 220b ⊕ 1 ∗ 120a
1*11b⊕1 ∗ 1a ⊕ 2 ∗ 44a ⊕ 1 ∗ 44b ⊕ 1 ∗ 220a ⊕ 1 ∗ 252a ⊕ 1 ∗ 320b ⊕ 1 ∗ 320a
3*11b⊕1 ∗ 11a ⊕ 1 ∗ 220b ⊕ 3 ∗ 120a ⊕ 1 ∗ 1242a
2*11a⊕1 ∗ 1a ⊕ 1 ∗ 44a ⊕ 1 ∗ 44b ⊕ 2 ∗ 252a
1*11b⊕2 ∗ 1a
2*11a⊕2 ∗ 44a ⊕ 1 ∗ 220b
1*11b⊕2 ∗ 44b ⊕ 2 ∗ 220a ⊕ 1 ∗ 320b
1*11b⊕1 ∗ 11a ⊕ 2 ∗ 1a ⊕ 1 ∗ 220b
1*11b⊕1 ∗ 11a ⊕ 1 ∗ 44a ⊕ 2 ∗ 252a
1*11b⊕1 ∗ 11a ⊕ 3 ∗ 120a
1*11a⊕1 ∗ 44a ⊕ 2 ∗ 44b ⊕ 1 ∗ 252a ⊕ 1 ∗ 320a
2*11b⊕1 ∗ 1a ⊕ 1 ∗ 120a
1*11a⊕1 ∗ 44a ⊕ 1 ∗ 252a
1*1a⊕1 ∗ 44b ⊕ 1 ∗ 220a
1*11b⊕1 ∗ 11a ⊕ 1 ∗ 1a ⊕ 1 ∗ 44a ⊕ 2 ∗ 220b
2*11b⊕1 ∗ 44b ⊕ 1 ∗ 220a
2*11a⊕1 ∗ 1a
1*44a⊕1 ∗ 252a
2*120a
1*44b⊕1 ∗ 252a
1*11b⊕1 ∗ 1a
1*11a⊕1 ∗ 44a
1*220a

Socle length 65

FIGURE 5.8. Socle series of P(220b)

1*220b
1*11b⊕1 ∗ 44b
1*11a⊕1 ∗ 1a ⊕ 1 ∗ 1242a
1*44a⊕1 ∗ 220a ⊕ 1 ∗ 252a ⊕ 1 ∗ 320a
2*11b⊕3 ∗ 1a ⊕ 1 ∗ 220b ⊕ 2 ∗ 120a ⊕ 1 ∗ 1792a
1*11b⊕2 ∗ 11a ⊕ 1 ∗ 44a ⊕ 2 ∗ 44b ⊕ 1 ∗ 252a ⊕ 1 ∗ 320b
2*11b⊕1 ∗ 11a ⊕ 2 ∗ 1a ⊕ 2 ∗ 44a ⊕ 1 ∗ 44b ⊕ 1 ∗ 120a ⊕ 1 ∗ 1242a
4*11a⊕1 ∗ 1a ⊕ 2 ∗ 44a ⊕ 1 ∗ 220b ⊕ 1 ∗ 220a ⊕ 1 ∗ 320a
3*11b⊕3 ∗ 1a ⊕ 3 ∗ 44b ⊕ 1 ∗ 220b ⊕ 1 ∗ 120a ⊕ 3 ∗ 220a ⊕ 1 ∗ 320b ⊕ 1 ∗ 1792a
3*11b⊕2 ∗ 11a ⊕ 4 ∗ 1a ⊕ 1 ∗ 44b ⊕ 2 ∗ 220b ⊕ 1 ∗ 120a ⊕ 1 ∗ 320b ⊕ 1 ∗ 1242a
3*11b⊕3 ∗ 11a ⊕ 2 ∗ 44a ⊕ 2 ∗ 120a ⊕ 4 ∗ 252a ⊕ 1 ∗ 320a
2*11b⊕5 ∗ 11a ⊕ 5 ∗ 1a ⊕ 1 ∗ 44b ⊕ 4 ∗ 120a ⊕ 2 ∗ 1242a ⊕ 1 ∗ 1792a
2*11a⊕5 ∗ 44a ⊕ 5 ∗ 44b ⊕ 1 ∗ 220b ⊕ 1 ∗ 220a ⊕ 1 ∗ 252a ⊕ 3 ∗ 320a
8*11b⊕4 ∗ 1a ⊕ 3 ∗ 44a ⊕ 1 ∗ 44b ⊕ 1 ∗ 220b ⊕ 2 ∗ 120a ⊕ 1 ∗ 1792a
1*11b⊕7 ∗ 11a ⊕ 3 ∗ 1a ⊕ 1 ∗ 44a ⊕ 2 ∗ 44b ⊕ 1 ∗ 220b ⊕ 2 ∗ 252a ⊕ 1 ∗ 320b
4*11b⊕1 ∗ 11a ⊕ 3 ∗ 1a ⊕ 2 ∗ 44a ⊕ 5 ∗ 44b ⊕ 5 ∗ 220a ⊕ 1 ∗ 252a ⊕ 1 ∗ 320b ⊕ 1 ∗ 1242a
3*11b⊕4 ∗ 11a ⊕ 6 ∗ 1a ⊕ 3 ∗ 44a ⊕ 5 ∗ 220b ⊕ 3 ∗ 120a
6*11b⊕3 ∗ 11a ⊕ 3 ∗ 1a ⊕ 1 ∗ 44a ⊕ 3 ∗ 44b ⊕ 2 ∗ 120a ⊕ 2 ∗ 220a ⊕ 5 ∗ 252a ⊕ 2 ∗ 320b
2*11b⊕5 ∗ 11a ⊕ 5 ∗ 1a ⊕ 1 ∗ 44a ⊕ 2 ∗ 44b ⊕ 1 ∗ 220b ⊕ 7 ∗ 120a ⊕ 3 ∗ 1242a
2*11b⊕4 ∗ 11a ⊕ 1 ∗ 1a ⊕ 7 ∗ 44a ⊕ 4 ∗ 44b ⊕ 1 ∗ 220b ⊕ 3 ∗ 252a ⊕ 4 ∗ 320a
11*11b⊕2 ∗ 11a ⊕ 5 ∗ 1a ⊕ 1 ∗ 44a ⊕ 2 ∗ 44b ⊕ 7 ∗ 120a ⊕ 2 ∗ 220a ⊕ 1 ∗ 1792a
7*11a⊕4 ∗ 1a ⊕ 2 ∗ 44a ⊕ 5 ∗ 44b ⊕ 1 ∗ 220b ⊕ 1 ∗ 220a ⊕ 4 ∗ 252a
6*11b⊕4 ∗ 1a ⊕ 4 ∗ 44a ⊕ 3 ∗ 44b ⊕ 1 ∗ 220b ⊕ 1 ∗ 120a ⊕ 3 ∗ 220a ⊕ 1 ∗ 252a
3*11b⊕10 ∗ 11a ⊕ 2 ∗ 1a ⊕ 4 ∗ 44a ⊕ 1 ∗ 44b ⊕ 5 ∗ 220b ⊕ 1 ∗ 252a ⊕ 1 ∗ 1242a
6*11b⊕2 ∗ 11a ⊕ 4 ∗ 1a ⊕ 1 ∗ 44a ⊕ 7 ∗ 44b ⊕ 7 ∗ 220a ⊕ 1 ∗ 252a ⊕ 3 ∗ 320b
2*11b⊕6 ∗ 11a ⊕ 9 ∗ 1a ⊕ 2 ∗ 44a ⊕ 5 ∗ 220b ⊕ 3 ∗ 120a ⊕ 2 ∗ 1242a
5*11b⊕3 ∗ 11a ⊕ 1 ∗ 1a ⊕ 5 ∗ 44a ⊕ 1 ∗ 44b ⊕ 1 ∗ 120a ⊕ 6 ∗ 252a ⊕ 1 ∗ 320b ⊕ 1 ∗ 320a
3*11b⊕5 ∗ 11a ⊕ 5 ∗ 1a ⊕ 1 ∗ 44b ⊕ 11 ∗ 120a ⊕ 2 ∗ 1242a ⊕ 1 ∗ 1792a
1*11a⊕5 ∗ 44a ⊕ 6 ∗ 44b ⊕ 2 ∗ 252a ⊕ 3 ∗ 320a
10*11b⊕4 ∗ 1a ⊕ 2 ∗ 44a ⊕ 5 ∗ 120a ⊕ 1 ∗ 1792a
8*11a⊕1 ∗ 1a ⊕ 2 ∗ 44a ⊕ 3 ∗ 44b ⊕ 1 ∗ 220b ⊕ 5 ∗ 252a
4*11b⊕4 ∗ 1a ⊕ 1 ∗ 44a ⊕ 7 ∗ 44b ⊕ 7 ∗ 220a
3*11b⊕7 ∗ 11a ⊕ 3 ∗ 1a ⊕ 4 ∗ 44a ⊕ 7 ∗ 220b
6*11b⊕2 ∗ 11a ⊕ 4 ∗ 1a ⊕ 1 ∗ 44a ⊕ 3 ∗ 44b ⊕ 3 ∗ 220a ⊕ 2 ∗ 252a ⊕ 4 ∗ 320b
1*11b⊕6 ∗ 11a ⊕ 4 ∗ 1a ⊕ 2 ∗ 220b ⊕ 2 ∗ 120a ⊕ 2 ∗ 1242a
2*11b⊕6 ∗ 44a ⊕ 1 ∗ 44b ⊕ 2 ∗ 252a ⊕ 2 ∗ 320a
3*11b⊕2 ∗ 11a ⊕ 2 ∗ 1a ⊕ 7 ∗ 120a ⊕ 1 ∗ 1242a ⊕ 1 ∗ 1792a
2*11a⊕1 ∗ 1a ⊕ 2 ∗ 44a ⊕ 3 ∗ 44b ⊕ 3 ∗ 252a
2*11b⊕3 ∗ 1a ⊕ 2 ∗ 120a
3*11a⊕2 ∗ 44a ⊕ 1 ∗ 44b ⊕ 1 ∗ 220b ⊕ 1 ∗ 252a
2*11b⊕1 ∗ 1a ⊕ 3 ∗ 44b ⊕ 3 ∗ 220a ⊕ 1 ∗ 320b
1*11b⊕3 ∗ 11a ⊕ 2 ∗ 1a ⊕ 1 ∗ 44a ⊕ 2 ∗ 220b
2*11b⊕1 ∗ 11a ⊕ 1 ∗ 1a ⊕ 1 ∗ 44a ⊕ 1 ∗ 220a ⊕ 2 ∗ 252a ⊕ 1 ∗ 320b
1*11b⊕2 ∗ 11a ⊕ 1 ∗ 1a ⊕ 1 ∗ 220b ⊕ 3 ∗ 120a ⊕ 1 ∗ 1242a
1*11a⊕2 ∗ 44a ⊕ 2 ∗ 44b ⊕ 1 ∗ 252a ⊕ 1 ∗ 320a
3*11b⊕1 ∗ 1a ⊕ 1 ∗ 120a
2*11a⊕1 ∗ 1a ⊕ 1 ∗ 44a ⊕ 1 ∗ 252a
1*1a⊕1 ∗ 44b ⊕ 1 ∗ 220a
1*11b⊕1 ∗ 11a ⊕ 1 ∗ 1a ⊕ 1 ∗ 44a ⊕ 2 ∗ 220b
2*11b⊕1 ∗ 11a ⊕ 1 ∗ 44b ⊕ 1 ∗ 220a ⊕ 1 ∗ 252a ⊕ 1 ∗ 320b
2*11a⊕2 ∗ 1a ⊕ 1 ∗ 120a
2*44a⊕1 ∗ 44b ⊕ 1 ∗ 252a ⊕ 1 ∗ 320a
3*11b⊕3 ∗ 120a
2*11a⊕1 ∗ 44b ⊕ 2 ∗ 252a
1*11b⊕2 ∗ 1a
2*11a⊕2 ∗ 44a ⊕ 1 ∗ 220b
1*11b⊕2 ∗ 44b ⊕ 2 ∗ 220a ⊕ 1 ∗ 320b
1*11a⊕2 ∗ 1a ⊕ 1 ∗ 220b
1*11b⊕1 ∗ 44a ⊕ 1 ∗ 252a
1*11b⊕1 ∗ 11a ⊕ 1 ∗ 120a
1*44a⊕1 ∗ 44b
1*11b⊕1 ∗ 120a
1*11a⊕1 ∗ 252a
1*1a⊕1 ∗ 220a
1*220b

Socle length 65

Figure 5.9. Socle series of P(252a)

$252a$
$1a \oplus 1 * 120a$
$11a \oplus 44a \oplus 44b \oplus 1 * 220b \oplus 1 * 320a$
$1a \oplus 4 * 11b \oplus 44a \oplus 44b \oplus 220a \oplus 1 * 320b$
$4*1a \oplus 3 * 11a \oplus 120a \oplus 1242a$
$11a \oplus 44a \oplus 44b \oplus 220a \oplus 3 * 252a \oplus 320a$
$4*1a \oplus 3 * 11b \oplus 3 * 120a \oplus 220b \oplus 1792a$
$3*11a \oplus 11b \oplus 2 * 44a \oplus 3 * 44b \oplus 120a \oplus 220b \oplus 2 * 252a$
$3*1a \oplus 2 * 11a \oplus 4 * 11b \oplus 3 * 44a \oplus 2 * 44b \oplus 120a \oplus 320b \oplus 1242a$
$2*1a \oplus 5 * 11a \oplus 3 * 44a \oplus 44b \oplus 120a \oplus 1 * 220b \oplus 2 * 320a$
$2*1a \oplus 6 * 11b \oplus 2 * 44a \oplus 4 * 44b \oplus 120a \oplus 4 * 220a \oplus 320b \oplus 1792a$
$7*1a \oplus 3 * 11a \oplus 3 * 11b \oplus 1 * 44b \oplus 2 * 120a \oplus 3 * 220b \oplus 252a \oplus 320b \oplus 1242a$
$1a \oplus 3 * 11a \oplus 4 * 11b \oplus 2 * 44a \oplus 44b \oplus 120a \oplus 220a \oplus 5 * 252a \oplus 320a$
$6*1a \oplus 6 * 11a \oplus 3 * 11b \oplus 44a \oplus 44b \oplus 5 * 120a \oplus 2 * 220b \oplus 2 * 1242a \oplus 1792a$
$1a \oplus 3 * 11a \oplus 3 * 11b \oplus 5 * 44a \oplus 7 * 44b \oplus 120a \oplus 220b \oplus 2 * 220a \oplus 2 * 252a \oplus 320b \oplus 2 * 320a$
$6*1a \oplus 3 * 11a \oplus 9 * 11b \oplus 3 * 44a \oplus 44b \oplus 5 * 120a \oplus 220a \oplus 220b \oplus 1242a$
$4*1a \oplus 9 * 11a \oplus 11b \oplus 3 * 44a \oplus 3 * 44b \oplus 5 * 252a \oplus 2 * 320a$
$6*1a \oplus 11a \oplus 6 * 11b \oplus 3 * 44a \oplus 5 * 44b \oplus 4 * 120a \oplus 5 * 220a \oplus 2 * 252a \oplus 320b \oplus 1242a \oplus 1792a$
$6*1a \oplus 6 * 11a \oplus 3 * 11b \oplus 4 * 44a \oplus 3 * 44b \oplus 2 * 120a \oplus 6 * 220b \oplus 2 * 252a \oplus 320a$
$5*1a \oplus 3 * 11a \oplus 10 * 11b \oplus 2 * 44a \oplus 4 * 44b \oplus 2 * 120a \oplus 3 * 220a \oplus 2 * 252a \oplus 2 * 320b$
$3*1a \oplus 11 * 11a \oplus 11b \oplus 2 * 44a \oplus 2 * 44b \oplus 3 * 120a \oplus 4 * 220b \oplus 252a \oplus 4 * 1242a$
$3*1a \oplus 11a \oplus 6 * 11b \oplus 7 * 44a \oplus 6 * 44b \oplus 4 * 220a \oplus 2 * 252a \oplus 3 * 320a \oplus 2 * 320b$
$10*1a \oplus 6 * 11a \oplus 9 * 11b \oplus 2 * 44a \oplus 44b \oplus 8 * 120a \oplus 220a \oplus 3 * 220b \oplus 1792a$
$3*1a \oplus 6 * 11a \oplus 3 * 11b \oplus 4 * 44a \oplus 6 * 44b \oplus 120a \oplus 220a \oplus 6 * 252a \oplus 320b$
$5*1a \oplus 3 * 11a \oplus 7 * 11b \oplus 2 * 44a \oplus 2 * 44b \oplus 8 * 120a \oplus 220a \oplus 220b \oplus 252a \oplus 2 * 1242a$
$9*11a \oplus 11b \oplus 6 * 44a \oplus 4 * 44b \oplus 2 * 220b \oplus 4 * 252a \oplus 3 * 320a$
$7*1a \oplus 11a \oplus 10 * 11b \oplus 44a \oplus 6 * 44b \oplus 4 * 120a \oplus 6 * 220a \oplus 2 * 320b \oplus 1792a$
$6*1a \oplus 6 * 11a \oplus 2 * 11b \oplus 3 * 44a \oplus 3 * 44b \oplus 5 * 220b \oplus 3 * 252a \oplus 1242a$
$4*1a \oplus 2 * 11a \oplus 8 * 11b \oplus 3 * 44a \oplus 2 * 44b \oplus 120a \oplus 2 * 220a \oplus 3 * 252a \oplus 320b$
$4*1a \oplus 11 * 11a \oplus 2 * 11b \oplus 3 * 44a \oplus 5 * 120a \oplus 4 * 220b \oplus 1 * 252a \oplus 2 * 1242a$
$3*1a \oplus 2 * 11a \oplus 4 * 11b \oplus 5 * 44a \oplus 8 * 44b \oplus 5 * 220a \oplus 2 * 252a \oplus 2 * 320b \oplus 2 * 320a$
$8*1a \oplus 5 * 11a \oplus 5 * 11b \oplus 2 * 44a \oplus 6 * 120a \oplus 3 * 220b \oplus 1792a$
$2*1a \oplus 4 * 11a \oplus 3 * 11b \oplus 5 * 44a \oplus 3 * 44b \oplus 6 * 252a \oplus 320a \oplus 320b$
$2*1a \oplus 3 * 11a \oplus 5 * 11b \oplus 2 * 44b \oplus 6 * 120a \oplus 2 * 220a \oplus 2 * 1242a$
$2*1a \oplus 5 * 11a \oplus 11b \oplus 5 * 44a \oplus 3 * 44b \oplus 3 * 220b \oplus 2 * 252a \oplus 320a$
$4*1a \oplus 11a \oplus 6 * 11b \oplus 2 * 44b \oplus 3 * 120a \oplus 2 * 220a \oplus 252a \oplus 2 * 320b \oplus 1792a$
$3*1a \oplus 6 * 11a \oplus 2 * 44a \oplus 44b \oplus 120a \oplus 2 * 220b \oplus 252a$
$1a \oplus 3 * 11b \oplus 3 * 44a \oplus 4 * 44b \oplus 3 * 220a \oplus 1 * 252a \oplus 320a \oplus 320b$
$2*1a \oplus 4 * 11a \oplus 4 * 11b \oplus 44a \oplus 3 * 120a \oplus 220b \oplus 1242a$
$1a \oplus 3 * 11a \oplus 11b \oplus 2 * 44a \oplus 2 * 44b \oplus 220a \oplus 4 * 252a \oplus 320b$
$5*1a \oplus 11a \oplus 2 * 11b \oplus 4 * 120a \oplus 220b$
$3*11a \oplus 11b \oplus 3 * 44a \oplus 2 * 44b \oplus 220b \oplus 252a \oplus 320a$
$1a \oplus 11a \oplus 4 * 11b \oplus 2 * 44b \oplus 120a \oplus 2 * 220a \oplus 320b \oplus 1242a$
$3*1a \oplus 3 * 11a \oplus 11b \oplus 2 * 44a \oplus 220b \oplus 252a$
$1a \oplus 11a \oplus 11b \oplus 44a \oplus 44b \oplus 220a \oplus 2 * 252a$
$2*1a \oplus 2 * 11a \oplus 2 * 11b \oplus 44a \oplus 3 * 120a \oplus 2 * 220b$
$2*11a \oplus 2 * 11b \oplus 44a \oplus 3 * 44b \oplus 220a \oplus 2 * 252a \oplus 320b \oplus 320a$
$3*1a \oplus 2 * 11a \oplus 3 * 11b \oplus 2 * 120a$
$1a \oplus 2 * 11a \oplus 3 * 44a \oplus 44b \oplus 2 * 252a \oplus 320a$
$1a \oplus 3 * 11b \oplus 44b \oplus 3 * 120a \oplus 220a$
$2*1a \oplus 2 * 11a \oplus 11b \oplus 44a \oplus 44b \oplus 2 * 220b \oplus 2 * 252a$
$2*1a \oplus 11a \oplus 3 * 11b \oplus 44b \oplus 220a \oplus 320b$
$1a \oplus 4 * 11a \oplus 2 * 44a \oplus 220b$
$11b \oplus 2 * 44a \oplus 2 * 44b \oplus 2 * 220a \oplus 252a$
$2*1a \oplus 2 * 11b \oplus 3 * 120a \oplus 220b$
$2*11a \oplus 11b \oplus 44b \oplus 4 * 252a$
$2*1a \oplus 11a \oplus 11b \oplus 2 * 120a$
$2*11a \oplus 2 * 44a \oplus 44b \oplus 320a$
$11b \oplus 44b \oplus 120a \oplus 220a \oplus 320b$
$220b \oplus 252a$
$1a \oplus 11b$
$2*11a \oplus 44a \oplus 220b$
$11b \oplus 44a \oplus 44b \oplus 220a$
$1a \oplus 120a$
$252a$

Socle length 65

Figure 5.10. Socle series of P(320a)

1 * 320a
1*11b⊕1 * 1792a
1*1a⊕1 * 320b
1*120a⊕1 * 252a ⊕ 1 * 1242a
1*1a⊕2 * 11a ⊕ 1 * 120a
2*44b⊕1 * 44a ⊕ 1 * 220a ⊕ 1 * 320a
3*11b⊕1 * 1a ⊕ 1 * 44a ⊕ 1 * 220b
1*11b⊕2 * 1a ⊕ 2 * 11a
1*11a⊕1 * 44b ⊕ 1 * 220a ⊕ 1 * 252a
1*11b⊕1 * 1a ⊕ 1 * 44a ⊕ 1 * 120a ⊕ 1 * 220a ⊕ 1 * 220b
1*11b⊕1 * 11a ⊕ 2 * 120a ⊕ 1 * 220b ⊕ 1 * 252a
1*11b⊕1 * 1a ⊕ 2 * 11a ⊕ 2 * 44b ⊕ 1 * 120a ⊕ 1 * 252a ⊕ 1 * 1242a
1*1a⊕1 * 11a ⊕ 1 * 44b ⊕ 2 * 44a ⊕ 2 * 320a
4*11b⊕1 * 1a ⊕ 1 * 11a ⊕ 3 * 44a ⊕ 1 * 120a ⊕ 1 * 1792a
1*11b⊕2 * 1a ⊕ 2 * 11a ⊕ 1 * 44b ⊕ 1 * 220a ⊕ 1 * 252a ⊕ 2 * 320b
1*11b⊕2 * 1a ⊕ 1 * 44b ⊕ 1 * 44a ⊕ 1 * 220a
1*11b⊕2 * 1a ⊕ 2 * 11a ⊕ 1 * 44a ⊕ 2 * 220b
3*11b⊕1 * 1a ⊕ 1 * 11a ⊕ 2 * 44b ⊕ 1 * 120a ⊕ 2 * 220a ⊕ 1 * 252a ⊕ 1 * 320b
2*1a⊕3 * 11a ⊕ 2 * 120a ⊕ 1 * 220b ⊕ 1 * 1242a
1*11b⊕1 * 11a ⊕ 1 * 44b ⊕ 2 * 44a ⊕ 1 * 320a ⊕ 2 * 252a
3*11b⊕2 * 1a ⊕ 1 * 11a ⊕ 4 * 120a ⊕ 1 * 1242a ⊕ 1 * 1792a
1*11a⊕3 * 44b ⊕ 1 * 44a ⊕ 1 * 320a ⊕ 2 * 252a
4*11b⊕3 * 1a ⊕ 1 * 44a ⊕ 1 * 120a
5*11a⊕1 * 44b ⊕ 2 * 44a ⊕ 1 * 220b ⊕ 1 * 252a
2*11b⊕2 * 1a ⊕ 4 * 44b ⊕ 1 * 44a ⊕ 3 * 220a ⊕ 1 * 320b
1*11b⊕3 * 1a ⊕ 2 * 11a ⊕ 1 * 44a ⊕ 1 * 120a ⊕ 3 * 220b
3*11b⊕1 * 1a ⊕ 2 * 11a ⊕ 1 * 44b ⊕ 1 * 220a ⊕ 2 * 252a ⊕ 1 * 320b
2*1a⊕3 * 11a ⊕ 2 * 120a ⊕ 1 * 220b ⊕ 2 * 1242a
1*11b⊕1 * 44b ⊕ 3 * 44a ⊕ 2 * 320a ⊕ 1 * 252a
3*11b⊕2 * 1a ⊕ 1 * 11a ⊕ 4 * 120a
1*1a⊕1 * 11a ⊕ 3 * 44b ⊕ 1 * 44a ⊕ 2 * 252a
3*11b⊕2 * 1a ⊕ 1 * 44a ⊕ 1 * 120a
4*11a⊕2 * 44a ⊕ 1 * 220b ⊕ 1 * 252a
1*11b⊕1 * 1a ⊕ 3 * 44b ⊕ 4 * 220a ⊕ 1 * 320b
1*11b⊕2 * 1a ⊕ 2 * 11a ⊕ 1 * 44a ⊕ 3 * 220b
3*11b⊕1 * 1a ⊕ 1 * 44a ⊕ 2 * 252a ⊕ 1 * 320b
1*11b⊕3 * 11a ⊕ 3 * 120a ⊕ 1 * 1242a
1*11a⊕1 * 44b ⊕ 2 * 44a ⊕ 1 * 320a ⊕ 1 * 252a
1*11b⊕2 * 1a ⊕ 2 * 120a ⊕ 1 * 1792a
1*11a⊕1 * 44b ⊕ 1 * 44a ⊕ 1 * 252a
1*11b⊕1 * 1a ⊕ 1 * 44b ⊕ 1 * 220a
1*11b⊕1 * 1a ⊕ 2 * 11a ⊕ 1 * 44a ⊕ 1 * 220b
1*11b⊕1 * 11a ⊕ 1 * 44b ⊕ 1 * 220a ⊕ 1 * 252a ⊕ 1 * 320b
2*1a⊕1 * 11a ⊕ 1 * 120a ⊕ 1 * 220b
1*11b⊕1 * 44b ⊕ 1 * 44a ⊕ 1 * 320a
2*11b⊕1 * 120a ⊕ 1 * 1242a
1*1a⊕1 * 11a ⊕ 1 * 252a
1*1a
1*11a⊕1 * 44a ⊕ 1 * 220b
1*11b⊕1 * 44b ⊕ 1 * 220a
1*1a⊕1 * 11a
1*44a⊕1 * 252a
1*11b⊕1 * 120a
1*44b
1*11b
1*11a
1*44b⊕1 * 220a
1*11b⊕1 * 1a
1*252a
1*120a
1*320a

Socle length 61

FIGURE 5.11. Socle series of P(320b)

$1 * 320b$

$1*120a \oplus 1 * 1242a$

$2*11a \oplus 1 * 1a \oplus 1 * 252a \oplus 1 * 320a$

$1*1a \oplus 1 * 11b \oplus 1 * 44b \oplus 1 * 44a \oplus 1 * 220a \oplus 1 * 1792a$

$1*11a \oplus 2 * 1a \oplus 1 * 11b \oplus 1 * 44a \oplus 2 * 220b$

$1*11a \oplus 1 * 1a \oplus 3 * 11b \oplus 1 * 44b \oplus 1 * 320b \oplus 1 * 252a$

$2*11a \oplus 1 * 1a \oplus 2 * 120a \oplus 1 * 1242a$

$1*11a \oplus 1 * 44b \oplus 2 * 44a \oplus 1 * 220a \oplus 1 * 320a$

$1*1a \oplus 3 * 11b \oplus 1 * 44a \oplus 2 * 120a \oplus 1 * 220b$

$2*11a \oplus 1 * 11b \oplus 2 * 44b \oplus 2 * 252a$

$1*11a \oplus 2 * 1a \oplus 1 * 11b \oplus 1 * 44b \oplus 1 * 44a \oplus 1 * 220a$

$3*11a \oplus 1 * 1a \oplus 1 * 11b \oplus 3 * 44a \oplus 2 * 220b$

$1*11a \oplus 2 * 1a \oplus 3 * 11b \oplus 2 * 44b \oplus 3 * 220a \oplus 2 * 320b \oplus 1 * 252a$

$2*11a \oplus 4 * 1a \oplus 3 * 120a \oplus 1 * 220b \oplus 2 * 1242a$

$1*11a \oplus 1 * 11b \oplus 1 * 44b \oplus 2 * 44a \oplus 2 * 252a \oplus 2 * 320a$

$1*11a \oplus 3 * 1a \oplus 4 * 11b \oplus 4 * 120a \oplus 1 * 1792a$

$2*11a \oplus 4 * 44b \oplus 1 * 44a \oplus 2 * 252a$

$2*1a \oplus 4 * 11b \oplus 2 * 44a \oplus 1 * 120a$

$5*11a \oplus 2 * 44a \oplus 1 * 220b \oplus 1 * 252a$

$3*1a \oplus 1 * 11b \oplus 5 * 44b \oplus 4 * 220a \oplus 1 * 320b$

$1*11a \oplus 3 * 1a \oplus 2 * 11b \oplus 1 * 44a \oplus 1 * 120a \oplus 4 * 220b$

$3*11a \oplus 1 * 1a \oplus 4 * 11b \oplus 1 * 320b \oplus 3 * 252a$

$4*11a \oplus 1 * 1a \oplus 3 * 120a \oplus 2 * 1242a$

$2*44b \oplus 4 * 44a \oplus 3 * 320a$

$2*1a \oplus 4 * 11b \oplus 4 * 120a \oplus 1 * 1792a$

$2*11a \oplus 1 * 1a \oplus 3 * 44b \oplus 3 * 252a$

$3*1a \oplus 3 * 11b \oplus 1 * 44a \oplus 1 * 220a$

$4*11a \oplus 3 * 44a \oplus 2 * 220b$

$1*1a \oplus 2 * 11b \oplus 3 * 44b \oplus 3 * 220a \oplus 2 * 320b$

$2*11a \oplus 3 * 1a \oplus 2 * 220b$

$2*11b \oplus 1 * 44a \oplus 2 * 252a$

$2*11a \oplus 1 * 11b \oplus 3 * 120a \oplus 1 * 1242a$

$1*11a \oplus 2 * 44b \oplus 2 * 44a \oplus 1 * 252a$

$2*1a \oplus 2 * 11b \oplus 2 * 120a$

$2*11a \oplus 1 * 44a \oplus 1 * 252a$

$1*1a \oplus 2 * 44b \oplus 2 * 220a$

$1*11a \oplus 1 * 1a \oplus 2 * 11b \oplus 1 * 44a \oplus 2 * 220b$

$1*11a \oplus 1 * 1a \oplus 2 * 11b \oplus 1 * 320b \oplus 1 * 252a$

$1*11a \oplus 1 * 1a \oplus 2 * 120a \oplus 1 * 1242a$

$1*44b \oplus 1 * 44a \oplus 2 * 320a$

$2*11b \oplus 1 * 120a \oplus 1 * 1792a$

$1*11a \oplus 1 * 1a \oplus 1 * 252a$

$1*1a$

$1*11a \oplus 1 * 44a \oplus 1 * 220b$

$1*11b \oplus 1 * 44b \oplus 1 * 220a \oplus 1 * 320b$

$1*11a \oplus 1 * 1a$

$1*44a \oplus 1 * 252a$

$1*11b \oplus 2 * 120a$

$1*11a \oplus 1 * 44b \oplus 1 * 252a$

$1*1a \oplus 1 * 11b$

$1*11a \oplus 1 * 44a$

$1*44b \oplus 1 * 220a$

$1*220b$

$1*11b$

$1*11a$

$1*44a$

$1*120a$

$1*252a$

$1*1a$

$1*11a$

$1*320b$

Socle length 61

FIGURE 5.12. Socle series of P(1242a)

$1242a$

$1a \oplus 11a \oplus 320a$

$11b \oplus 44a \oplus 44b \oplus 220a \oplus 1792a$

$1a \oplus 11b \oplus 220b \oplus 320b$

$11a \oplus 11b \oplus 120a \oplus 252a$

$2*11a \oplus 44b \oplus 120a \oplus 1242a$

$44a \oplus 44b \oplus 220a \oplus 320a$

$2*1a \oplus 2*11b \oplus 44a \oplus 120a \oplus 220b$

$11a \oplus 11b \oplus 44b \oplus 252a$

$1a \oplus 11a \oplus 11b \oplus 44a \oplus 44b \oplus 220a \oplus 1242a$

$2*1a \oplus 2*11a \oplus 11b \oplus 2*44a \oplus 220b$

$2*1a \oplus 11a \oplus 2*11b \oplus 44b \oplus 2*120a \oplus 2*220a \oplus 1*252a \oplus 320b$

$3*1a \oplus 2*11a \oplus 44b \oplus 2*120a \oplus 220b \oplus 1242a$

$11a \oplus 2*11b \oplus 3*44a \oplus 44b \oplus 2*252a \oplus 2*320a$

$2*1a \oplus 2*11a \oplus 3*11b \oplus 2*120a \oplus 1792a$

$1a \oplus 2*11a \oplus 2*44a \oplus 2*44b \oplus 220a \oplus 220b \oplus 252a$

$1*1a \oplus 3*11b \oplus 44a \oplus 44b \oplus 120a \oplus 220b$

$1a \oplus 4*11a \oplus 11b \oplus 44a \oplus 44b \oplus 220b \oplus 252a$

$2*1a \oplus 11a \oplus 2*11b \oplus 44a \oplus 3*44b \oplus 3*220a \oplus 320b$

$3*1a \oplus 2*11a \oplus 2*11b \oplus 2*44a \oplus 2*220b \oplus 2*120a$

$1*1a \oplus 2*11a \oplus 2*11b \oplus 1*44a \oplus 4*252a \oplus 1*320b$

$2*11a \oplus 2*1a \oplus 1*11b \oplus 4*120a \oplus 2*1242a$

$11a \oplus 3*44a \oplus 3*44b \oplus 1*220b \oplus 2*320a$

$2*1a \oplus 6*11b \oplus 44b \oplus 2*120a \oplus 220a \oplus 1792a$

$1a \oplus 3*11a \oplus 2*44b \oplus 2*252a$

$2*1a \oplus 2*11b \oplus 2*44a \oplus 2*44b \oplus 2*220a$

$2*1a \oplus 4*11a \oplus 11b \oplus 2*44a \oplus 2*220b$

$2*1a \oplus 11a \oplus 2*11b \oplus 2*44b \oplus 2*220a \oplus 1*252a \oplus 2*320b$

$2*1a \oplus 11a \oplus 2*120a \oplus 2*220b \oplus 1*1242a$

$11a \oplus 2*11b \oplus 44a \oplus 44b \oplus 2*252a$

$1a \oplus 2*11a \oplus 3*120a \oplus 1242a$

$2*44a \oplus 2*44b \oplus 320a$

$1a \oplus 3*11b \oplus 2*120a$

$2*11a \oplus 44b \oplus 2*252a$

$2*1a \oplus 11b \oplus 44a \oplus 44b \oplus 2*220a$

$2*11a \oplus 11b \oplus 2*44a \oplus 2*220b$

$1a \oplus 2*11b \oplus 44b \oplus 220a \oplus 320b$

$1a \oplus 2*11a \oplus 120a \oplus 220b \oplus 1242a$

$11b \oplus 44a \oplus 252a \oplus 320a$

$1a \oplus 11a \oplus 120a \oplus 1792a$

$44a \oplus 44b$

$11b \oplus 120a$

$11a \oplus 252a$

$1a \oplus 44b \oplus 220a$

$11a \oplus 44a \oplus 220b$

$1a \oplus 11b \oplus 320b$

$1242a$

Socle length 47

Figure 5.13. Socle series of P(1792a)

$1792a$

$320b$

$120a \oplus 1242a$

$1a \oplus 2 * 11a \oplus 320a$

$11b \oplus 44a \oplus 44b \oplus 220a$

$2 * 1a \oplus 11b \oplus 220b$

$11a \oplus 11b \oplus 252a$

$11a \oplus 120a$

$44a \oplus 44b \oplus 220a$

$11b \oplus 44a \oplus 120a \oplus 220b$

$11a \oplus 11b \oplus 44b \oplus 252a$

$1a \oplus 11a \oplus 44b \oplus 220a$

$1a \oplus 11a \oplus 11b \oplus 2 * 44a \oplus 220b$

$1a \oplus 11a \oplus 2 * 11b \oplus 220a \oplus 252a \oplus 320b$

$2 * 1a \oplus 11a \oplus 2 * 120a \oplus 1242a$

$11a \oplus 44a \oplus 44b \oplus 320a$

$1a \oplus 2 * 11b \oplus 120a \oplus 1792a$

$11a \oplus 44b \oplus 252a$

$1a \oplus 11b \oplus 44a$

$2 * 11a \oplus 44a \oplus 220b$

$1a \oplus 11b \oplus 2 * 44b \oplus 220a \oplus 320b$

$1a \oplus 120a \oplus 220b$

$11a \oplus 11b \oplus 252a$

$11a \oplus 120a \oplus 1242a$

$44a \oplus 44b \oplus 320a$

$1a \oplus 2 * 11b \oplus 120a$

$11a \oplus 44b \oplus 252a$

$1a \oplus 11b \oplus 44a \oplus 220a$

$11a \oplus 44a \oplus 220b$

$1a \oplus 11b \oplus 44b \oplus 220a$

$1a \oplus 11a \oplus 220b$

$11b \oplus 252a$

$11a \oplus 120a$

$44a \oplus 44b$

$11b \oplus 120a$

$11a \oplus 252a$

$1a \oplus 44b \oplus 220a$

$11a \oplus 11b \oplus 44a \oplus 220b$

$1a \oplus 11b \oplus 320b$

$120a \oplus 1242a$

$320a$

$1792a$

Socle length 42

Table 5.4. Correspondence between the composition factors of the projective indecomposable $e_H\mathbb{F}_2M_{24}e_H$-modules and the composition factors of the projective indecomposable $\mathbb{F}_2M_{24}$-modules

Simples of M_{24}			1a	11a	11b	44a	44b	120a	220a	220b	252a	320a	320b	1242a	1792a
P.I.M.s M_{24}	Comp. name	Dir. name													
P(1a)	npim1a	npim1a	1a	1c	1b	2a	2b	6b	6c	6a	8a	8c	8b	46a	64a
P(11a)	npim11a	npim1b	1c	1a	1b	2a	2b	6a	6b	6c	8a	8c	8b	46a	64a
P(11b)	npim11b	npim1c	1c	1b	1a	2a	2b	6a	6b	6c	8a	8c	8b	46a	64a
P(44a)	npim44a	npim2a	1c	1b	1a	2a	2b	6a	6b	6c	8a	8c	8b	46a	64a
P(44b)	npim44b	npim2b	1a	1b	1c	2b	2a	6a	6b	6c	8a	8c	8b	46a	64a
P(120a)	npim120a	npim6a	1c	1a	1b	2a	2b	6a	6b	6c	8a	8b	8c	46a	64a
P(220a)	npim220b	npim6b	1c	1b	1a	2a	2b	6c	6a	6b	8a	8c	8b	46a	64a
P(220b)	npim220c	npim6c	1c	1b	1a	2a	2b	6b	6c	6a	8a	8c	8b	46a	64a
P(252a)	npim252c	npim8c	1a	1b	1c	2a	2b	6a	6c	6b	8a	8c	8b	46a	64a
P(320a)	npim320b	npim8b	1b	1c	1a	2b	2a	6a	6b	6c	8b	8a	8c	46a	64a
P(320b)	npim320c	npim8a	1b	1a	1c	2b	2a	6a	6b	6c	8b	8c	8a	46a	64a
P(1242a)	npim1242a	mpim46a	1b	1a	1c	2a	2b	6c	6a	6b	8b	8a	8c	46a	64a
P(1792a)	npim1792a	mpim64a	1b	1a	1c	2a	2b	6a	6b	6c	8c	8b	8a	46a	64a

5.2 The P.I.M.s of A_{12} in characteristic 2

In this section we compute the projective indecomposable modules of the alternating group A_{12} over the field $\mathbb{F}_4$ with 4 elements. We will also give the socle series of the P.I.M.s of A_{12}. The details of most of the work can be shown with the techniques we described in detail in the last section, hence they will be omitted in this section. Let $G = A_{12}$ and $\mathbb{F} = \mathbb{F}_4$ throughout the section.

We will use the standard generators `g1,g2` for A_{12}, where

$$\texttt{g1} := (1, 2, 3),$$

$$\texttt{g2} := (2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12).$$

We begin by reviewing the main facts about the simple $\mathbb{F}G$-modules.

Lemma 5.2.1. *There are* 18 *simple* $\mathbb{F}G$*-modules, which we denote by* $1a, 10a, 16a, 16b, 44a, 100a, 144a, 144b, 164a, 320a, 416a, 570a, 1046a, 1184a, 1184b, 1408a, 1792a, 5632a$. *The splitting field for* A_{12} *in characteristic* 2 *is* $\mathbb{F}_4$.

We continue by identifying the subgroup H of G we are going to use for faithful condensation.

Lemma 5.2.2. *Let* H *be a subgroup of* A_{12} *of order* 21 *that is in the* 389*th conjugacy class of the table of marks of* A_{12}*. Then* e_H *is a faithful idempotent of* $\mathbb{F}G$*. The condensation subalgebra* $e_H\mathbb{F}Ge_H$ *has dimension* 546000*. Moreover the following table gives the dimension of the condensed simple* $\mathbb{F}G$*-modules.*

TABLE 5.5. Dimensions of the simple $\mathbb{F}_4A_{12}$-modules and the simple $e_H\mathbb{F}_4A_{12}e_H$-modules

Dimensions of the simple $\mathbb{F}G$-modules																	
1	10	16	16	44	100	144	144	164	320	416	570	1046	1184	1184	1408	1792	5632
Dimensions of the simple $e_H\mathbb{F}Ge_H$-modules																	
1	4	4	4	6	2	16	16	8	26	16	18	32	52	52	68	80	248

We proceed by listing the dimensions of the projective indecomposable modules of A_{12} and the dimensions of the projective indecomposable $e_H\mathbb{F}Ge_H$-modules.

Lemma 5.2.3. *There are* 18 *projective indecomposable modules of* A_{12} *with the following dimensions:*

Table 5.6. Dimensions of the projective indecomposable $\mathbb{F}_4A_{12}$-modules and the dimensions of the projective indecomposable $e_H\mathbb{F}_4A_{12}e_H$-modules

P.I.M.s	P(1a)	P(10a)	P(16a)	P(16b)	P(44a)	P(100a)	P(144a)	P(144b)	P(164a)
Dim.	204288	159232	72704	72704	69120	116224	27648	27648	59904
Cond. Dim.	9856	7680	3480	3480	3392	5600	1384	1384	2912

P.I.M.s	P(320a)	P(416a)	P(570a)	P(1046a)	P(1184a)	P(1184b)	P(1408a)	P(1792a)	P(5632a)
Dim.	7680	50688	48640	45056	22528	22528	6656	5632	5632
Cond. Dim.	400	2448	2360	2104	1080	1080	336	280	248

The next step is finding a generating set for the condensation subalgebra $e_H\mathbb{F}Ge_H$.

Lemma 5.2.4. *Let* N *be the representative subgroup of the* 1646*th conjugacy class of the table of marks with generators* a, b*. The faithful condensation subgroup* H *that is in the* 389*th conjugacy class of the table of marks of* A_{12} *is generated by*

$$\{ab^2a^{-1}, b^{-2}ab^2a^{-1}\}$$

and is normal in N*. The condensation subalgebra* $e_H\mathbb{F}Ge_H$ *is generated by the words* $e_Hz_3e_H, e_Hz_4e_H, e_Hz_5e_H, e_Hae_H, e_Hbe_H$.

Lemma 5.2.5. *Let* K_1 *be the subgroup representative of the* 994*th conjugacy class of the table of marks of* A_{12}*. The permutation module of* G *on the cosets of* K_1 *is projective and splits as:*

$$1_{K_1}^G = P(1a) \oplus 2 * P(10a) \oplus 2 * P(44a) \oplus 2 * P(100a) \oplus 4 * P(144a) \oplus 4 * P(144b) \oplus$$

$$4 * P(164a) \oplus 10 * P(320a) \oplus 4 * P(416a) \oplus 6 * P(570a) \oplus 12 * P(1046a) \oplus$$

$$12 * P(1184a) \oplus 12 * P(1184b) \oplus 28 * P(1408a) \oplus 24 * P(1792a) \oplus 84 * P(5632a)$$

The dimension of the $e_H \mathbb{F} G e_H$*-module* $1_{K_1}^G e_H$ *is* 182000.

Let K_2 *be the subgroup representative of the* 1243*th conjugacy class of the table of marks of* A_{12}. *The permutation module of* G *on the cosets of* K_2 *is projective and splits as:*

$$1_{K_2}^G = P(1a) \oplus P(16a) \oplus P(16b) \oplus 2 * P(44a) \oplus 2 * P(100a) \oplus 3 * P(144a) \oplus 3 * P(144b) \oplus$$

$$4 * P(320a) \oplus 6 * P(416a) \oplus 4 * P(570a) \oplus 8 * P(1046a) \oplus 13 * P(1184a) \oplus$$

$$13 * P(1184b) \oplus 20 * P(1408a) \oplus 20 * P(1792a) \oplus 62 * P(5632a)$$

The dimension of the $e_H \mathbb{F} G e_H$*-module* $1_{K_2}^G e_H$ *is* 141440.

With these two permutation modules we are able to construct the projective indecomposable modules of A_{12} and their socle series which are displayed in the following pages.

Figure 5.14. Socle series of $P(5632a)$

1*5632a
Socle length 1

Figure 5.15. Socle series of $P(1792a)$)

1*1792a
1*320a
1*1408a
1*320a
1*1792a
Socle length 5

Figure 5.16. Socle series of $P(1408a)$

1*1408a
1*320a$\oplus 1 * 1408a$
1*1792a
1*320a
1*1408a
Socle length 5

Figure 5.17. Socle series of $P(320a)$

1*320a
1*1408a$\oplus 1 * 1792a$
2*320a
1*1408a$\oplus 1 * 1792a$
1*320a
Socle length 5

FIGURE 5.18. Socle series of $P(1184b)$

$1*1184b$
$1*10a \oplus 1*16b \oplus 1*416a$
$1*1a \oplus 1*100a \oplus 1*144a$
$2*10a \oplus 1*16b$
$1*100a \oplus 1*44a \oplus 1*1046a \oplus 1*1184a$
$2*1a \oplus 2*10a \oplus 2*16a$
$1*100a \oplus 1*164a \oplus 1*416a \oplus 1*1184b$
$2*1a \oplus 3*10a \oplus 1*16a$
$2*100a \oplus 2*44a \oplus 1*570a \oplus 1*1046a \oplus 1*1184b$
$3*1a \oplus 3*10a \oplus 3*16b \oplus 1*144b$
$1*100a \oplus 1*164a \oplus 2*416a \oplus 1*1184a$
$3*1a \oplus 3*10a \oplus 1*16b$
$2*100a \oplus 2*44a \oplus 1*570a \oplus 1*1046a \oplus 1*1184a$
$2*1a \oplus 3*10a \oplus 2*16a \oplus 1*144a$
$1*100a \oplus 1*164a \oplus 1*416a \oplus 1*1184b$
$2*1a \oplus 2*10a \oplus 1*16a$
$1*100a \oplus 1*44a \oplus 1*1046a \oplus 1*1184b$
$1*1a \oplus 2*10a \oplus 2*16b$
$1*100a \oplus 1*416a$
$1*10a$
$1*1184a$
$1*10a$
$1*100a$
$1*10a$
$1*1184b$

Socle length 25

FIGURE 5.19. Socle series of $P(1184a)$

$1*1184a$
$1*10a \oplus 1*16a \oplus 1*416a$
$1*1a \oplus 1*100a \oplus 1*144b$
$2*10a \oplus 1*16a$
$1*100a \oplus 1*44a \oplus 1*1046a \oplus 1*1184b$
$2*1a \oplus 2*10a \oplus 2*16b$
$1*100a \oplus 1*164a \oplus 1*416a \oplus 1*1184a$
$2*1a \oplus 3*10a \oplus 1*16b$
$2*100a \oplus 2*44a \oplus 1*570a \oplus 1*1046a \oplus 1*1184a$
$3*1a \oplus 3*10a \oplus 3*16a \oplus 1*144a$
$1*100a \oplus 1*164a \oplus 2*416a \oplus 1*1184b$
$3*1a \oplus 3*10a \oplus 1*16a$
$2*100a \oplus 2*44a \oplus 1*570a \oplus 1*1046a \oplus 1*1184b$
$2*1a \oplus 3*10a \oplus 2*16b \oplus 1*144b$
$1*100a \oplus 1*164a \oplus 1*416a \oplus 1*1184a$
$2*1a \oplus 2*10a \oplus 1*16b$
$1*100a \oplus 1*44a \oplus 1*1046a \oplus 1*1184a$
$1*1a \oplus 2*10a \oplus 2*16a$
$1*100a \oplus 1*416a$
$1*10a$
$1*1184b$
$1*10a$
$1*100a$
$1*10a$
$1*1184a$

Socle length 25

FIGURE 5.20. Socle series of $P(1046a)$

$1*1046\mathrm{a}$

$1*1\mathrm{a} \oplus 1 * 16a \oplus 1 * 10a \oplus 1 * 16b$

$1*1\mathrm{a} \oplus 2 * 100a \oplus 1 * 416a \oplus 1 * 570a$

$1*1\mathrm{a} \oplus 1 * 16a \oplus 2 * 10a \oplus 1 * 16b \oplus 1 * 164a \oplus 1 * 144b \oplus 1 * 144a$

$1*1\mathrm{a} \oplus 1 * 100a \oplus 1 * 44a \oplus 2 * 1046a \oplus 1 * 1184a \oplus 1 * 1184b$

$3*1\mathrm{a} \oplus 2 * 16a \oplus 4 * 10a \oplus 2 * 16b \oplus 1 * 44a \oplus 1 * 164a$

$3*1\mathrm{a} \oplus 5 * 100a \oplus 1 * 164a \oplus 2 * 416a \oplus 1 * 570a$

$4*1\mathrm{a} \oplus 1 * 100a \oplus 2 * 16a \oplus 4 * 10a \oplus 2 * 16b \oplus 1 * 164a \oplus 1 * 416a \oplus 1 * 570a$

$2*1\mathrm{a} \oplus 1 * 100a \oplus 1 * 16a \oplus 1 * 10a \oplus 1 * 16b \oplus 3 * 44a \oplus 1 * 164a \oplus 1 * 144b \oplus 1 * 144a \oplus 1 * 570a \oplus 3 * 1046a \oplus 1 * 1184a \oplus 1 * 1184b$

$4*1\mathrm{a} \oplus 1 * 100a \oplus 2 * 16a \oplus 5 * 10a \oplus 2 * 16b \oplus 2 * 44a \oplus 1 * 164a \oplus 1 * 144b \oplus 1 * 144a \oplus 1 * 1046a$

$4*1\mathrm{a} \oplus 5 * 100a \oplus 1 * 16a \oplus 1 * 10a \oplus 1 * 16b \oplus 1 * 44a \oplus 1 * 164a \oplus 2 * 416a \oplus 1 * 570a$

$5*1\mathrm{a} \oplus 2 * 100a \oplus 1 * 16a \oplus 4 * 10a \oplus 1 * 16b \oplus 2 * 164a \oplus 1 * 416a \oplus 2 * 570a$

$4*1\mathrm{a} \oplus 1 * 100a \oplus 1 * 16a \oplus 1 * 10a \oplus 1 * 16b \oplus 2 * 44a \oplus 1 * 164a \oplus 1 * 416a \oplus 1 * 144b \oplus 1 * 144a \oplus 2 * 1046a \oplus 1 * 1184a \oplus 1 * 1184b$

$2*1\mathrm{a} \oplus 1 * 100a \oplus 2 * 16a \oplus 4 * 10a \oplus 2 * 16b \oplus 2 * 44a \oplus 1 * 164a \oplus 1 * 570a$

$3*1\mathrm{a} \oplus 3 * 100a \oplus 1 * 44a \oplus 1 * 416a \oplus 1 * 144b \oplus 1 * 144a \oplus 2 * 1046a$

$2*1\mathrm{a} \oplus 2 * 16a \oplus 4 * 10a \oplus 2 * 16b \oplus 1 * 44a \oplus 1 * 164a \oplus 1 * 416a \oplus 1 * 570a$

$4*1\mathrm{a} \oplus 3 * 100a \oplus 1 * 164a \oplus 1 * 570a \oplus 1 * 1184a \oplus 1 * 1184b$

$2*1\mathrm{a} \oplus 1 * 100a \oplus 1 * 16a \oplus 2 * 10a \oplus 1 * 16b \oplus 1 * 164a \oplus 1 * 416a \oplus 1 * 570a$

$2*1\mathrm{a} \oplus 1 * 16a \oplus 1 * 10a \oplus 1 * 16b \oplus 1 * 164a \oplus 1 * 144b \oplus 1 * 144a \oplus 1 * 1046a$

$1*100\mathrm{a} \oplus 2 * 44a \oplus 1 * 1046a$

$2*1\mathrm{a} \oplus 1 * 16a \oplus 1 * 10a \oplus 1 * 16b$

$1*100\mathrm{a} \oplus 1 * 164a \oplus 1 * 416a \oplus 1 * 570a$

$2*1\mathrm{a} \oplus 1 * 100a$

$1*16\mathrm{a} \oplus 1 * 10a \oplus 1 * 16b$

$1*1046\mathrm{a}$

Socle length 25

FIGURE 5.21. Socle series of $P(570a)$

$1*570\mathrm{a}$

$2*1\mathrm{a} \oplus 1 * 144b \oplus 1 * 144a$

$1*44\mathrm{a} \oplus 1 * 164a \oplus 1 * 570a \oplus 1 * 1046a$

$3*1\mathrm{a} \oplus 1 * 100a \oplus 1 * 16a \oplus 1 * 10a \oplus 1 * 16b$

$1*16\mathrm{a} \oplus 1 * 10a \oplus 1 * 16b \oplus 1 * 44a \oplus 2 * 164a \oplus 1 * 416a \oplus 2 * 570a$

$7*1\mathrm{a} \oplus 3 * 100a \oplus 1 * 10a \oplus 2 * 144b \oplus 2 * 144a$

$1*100\mathrm{a} \oplus 1 * 16a \oplus 1 * 10a \oplus 1 * 16b \oplus 3 * 44a \oplus 2 * 164a \oplus 1 * 416a \oplus 2 * 570a \oplus 1 * 1046a$

$7*1\mathrm{a} \oplus 2 * 100a \oplus 1 * 16a \oplus 3 * 10a \oplus 1 * 16b \oplus 1 * 144b \oplus 1 * 144a \oplus 1 * 1046a$

$1*100\mathrm{a} \oplus 2 * 16a \oplus 2 * 10a \oplus 2 * 16b \oplus 3 * 44a \oplus 3 * 164a \oplus 2 * 416a \oplus 2 * 570a \oplus 1 * 1046a \oplus 1 * 1184a \oplus 1 * 1184b$

$10*1\mathrm{a} \oplus 4 * 100a \oplus 1 * 16a \oplus 4 * 10a \oplus 1 * 16b \oplus 1 * 144b \oplus 1 * 144a$

$3*100\mathrm{a} \oplus 2 * 16a \oplus 2 * 10a \oplus 2 * 16b \oplus 3 * 44a \oplus 3 * 164a \oplus 2 * 416a \oplus 3 * 570a \oplus 1 * 1046a$

$9*1\mathrm{a} \oplus 3 * 100a \oplus 1 * 16a \oplus 4 * 10a \oplus 1 * 16b \oplus 2 * 144b \oplus 2 * 144a \oplus 2 * 1046a$

$1*100\mathrm{a} \oplus 3 * 16a \oplus 3 * 10a \oplus 3 * 16b \oplus 4 * 44a \oplus 4 * 164a \oplus 1 * 416a \oplus 3 * 570a \oplus 1 * 1184a \oplus 1 * 1184b$

$11*1\mathrm{a} \oplus 4 * 100a \oplus 2 * 10a \oplus 1 * 144b \oplus 1 * 144a \oplus 1 * 1046a$

$1*100\mathrm{a} \oplus 3 * 16a \oplus 3 * 10a \oplus 3 * 16b \oplus 2 * 44a \oplus 3 * 164a \oplus 2 * 416a \oplus 4 * 570a$

$7*1\mathrm{a} \oplus 4 * 100a \oplus 2 * 144b \oplus 2 * 144a \oplus 1 * 1046a$

$1*16\mathrm{a} \oplus 2 * 10a \oplus 1 * 16b \oplus 3 * 44a \oplus 2 * 164a \oplus 1 * 416a \oplus 1 * 570a \oplus 1 * 1046a$

$6*1\mathrm{a} \oplus 1 * 100a \oplus 1 * 16a \oplus 1 * 10a \oplus 1 * 16b \oplus 1 * 144b \oplus 1 * 144a \oplus 1 * 1046a$

$1*100\mathrm{a} \oplus 1 * 16a \oplus 1 * 10a \oplus 1 * 16b \oplus 2 * 44a \oplus 2 * 164a \oplus 1 * 416a \oplus 2 * 570a$

$6*1\mathrm{a} \oplus 1 * 100a$

$1*16\mathrm{a} \oplus 1 * 10a \oplus 1 * 16b \oplus 1 * 44a \oplus 1 * 164a \oplus 1 * 416a \oplus 2 * 570a$

$2*1\mathrm{a} \oplus 1 * 100a \oplus 1 * 144b \oplus 1 * 144a \oplus 1 * 1046a$

$1*44\mathrm{a} \oplus 1 * 164a$

$2*1\mathrm{a}$

$1*570\mathrm{a}$

Socle length 25

FIGURE 5.22. Socle series of $P(144b)$

$1*144b$

$1*16a \oplus 1*416a \oplus 1*570a$

$2*1a \oplus 1*144b \oplus 2*144a \oplus 1*1184a$

$1*16a \oplus 1*44a \oplus 1*164a \oplus 1*416a \oplus 1*1046a$

$2*1a \oplus 1*100a \oplus 1*16b \oplus 1*10a \oplus 1*144b \oplus 1*144a$

$1*16a \oplus 1*16b \oplus 1*10a \oplus 1*44a \oplus 1*164a \oplus 1*416a \oplus 2*570a$

$5*1a \oplus 2*100a \oplus 1*10a \oplus 2*144b$

$1*100a \oplus 1*16a \oplus 1*16b \oplus 1*10a \oplus 1*44a \oplus 2*164a \oplus 1*416a \oplus 1*570a$

$4*1a \oplus 2*100a \oplus 2*10a \oplus 1*144b \oplus 1*144a \oplus 1*1046a$

$2*16a \oplus 1*16b \oplus 1*10a \oplus 3*44a \oplus 1*164a \oplus 1*570a \oplus 1*1046a \oplus 1*1184b$

$6*1a \oplus 2*100a \oplus 1*16a \oplus 1*16b \oplus 3*10a$

$2*100a \oplus 1*16a \oplus 1*16b \oplus 1*10a \oplus 1*44a \oplus 2*164a \oplus 2*416a \oplus 2*570a$

$5*1a \oplus 2*100a \oplus 1*10a \oplus 1*144b \oplus 1*144a \oplus 1*1046a$

$1*16a \oplus 2*16b \oplus 2*10a \oplus 2*44a \oplus 2*164a \oplus 1*570a \oplus 1*1184a$

$4*1a \oplus 1*100a \oplus 1*144b \oplus 1*144a \oplus 1*1046a$

$1*16a \oplus 1*16b \oplus 1*10a \oplus 1*44a \oplus 1*164a \oplus 1*416a \oplus 2*570a$

$4*1a \oplus 2*100a \oplus 1*144a$

$1*16a \oplus 1*16b \oplus 1*10a \oplus 1*44a \oplus 1*164a \oplus 1*416a \oplus 1*570a$

$2*1a \oplus 1*100a \oplus 1*144b \oplus 1*144a \oplus 1*1046a$

$1*16a \oplus 1*44a \oplus 1*164a$

$2*1a$

$1*416a \oplus 1*570a$

$1*144b$

Socle length 23

FIGURE 5.23. Socle series of $P(144a)$

$1*144a$

$1*16b \oplus 1*416a \oplus 1*570a$

$2*1a \oplus 1*144a \oplus 2*144b \oplus 1*1184b$

$1*16b \oplus 1*44a \oplus 1*164a \oplus 1*416a \oplus 1*1046a$

$2*1a \oplus 1*100a \oplus 1*16a \oplus 1*10a \oplus 1*144a \oplus 1*144b$

$1*16b \oplus 1*16a \oplus 1*10a \oplus 1*44a \oplus 1*164a \oplus 1*416a \oplus 2*570a$

$5*1a \oplus 2*100a \oplus 1*10a \oplus 2*144a$

$1*100a \oplus 1*16b \oplus 1*16a \oplus 1*10a \oplus 1*44a \oplus 2*164a \oplus 1*416a \oplus 1*570a$

$4*1a \oplus 2*100a \oplus 2*10a \oplus 1*144a \oplus 1*144b \oplus 1*1046a$

$2*16b \oplus 1*16a \oplus 1*10a \oplus 3*44a \oplus 1*164a \oplus 1*570a \oplus 1*1046a \oplus 1*1184a$

$6*1a \oplus 2*100a \oplus 1*16b \oplus 1*16a \oplus 3*10a$

$2*100a \oplus 1*16b \oplus 1*16a \oplus 1*10a \oplus 1*44a \oplus 2*164a \oplus 2*416a \oplus 2*570a$

$5*1a \oplus 2*100a \oplus 1*10a \oplus 1*144a \oplus 1*144b \oplus 1*1046a$

$1*16b \oplus 2*16a \oplus 2*10a \oplus 2*44a \oplus 2*164a \oplus 1*570a \oplus 1*1184b$

$4*1a \oplus 1*100a \oplus 1*144a \oplus 1*144b \oplus 1*1046a$

$1*16b \oplus 1*16a \oplus 1*10a \oplus 1*44a \oplus 1*164a \oplus 1*416a \oplus 2*570a$

$4*1a \oplus 2*100a \oplus 1*144b$

$1*16b \oplus 1*16a \oplus 1*10a \oplus 1*44a \oplus 1*164a \oplus 1*416a \oplus 1*570a$

$2*1a \oplus 1*100a \oplus 1*144a \oplus 1*144b \oplus 1*1046a$

$1*16b \oplus 1*44a \oplus 1*164a$

$2*1a$

$1*416a \oplus 1*570a$

$1*144a$

Socle length 23

FIGURE 5.24. Socle series of $P(416a)$

1*416a

1*1a$\oplus 1 * 144a \oplus 1 * 144b \oplus 1 * 1184b \oplus 1 * 1184a$

1*16a$\oplus 1 * 10a \oplus 1 * 16b \oplus 1 * 44a \oplus 2 * 416a \oplus 1 * 1046a$

3*1a$\oplus 1 * 16a \oplus 2 * 10a \oplus 1 * 16b \oplus 1 * 144a \oplus 1 * 144b$

3*100a$\oplus 1 * 164a \oplus 2 * 416a \oplus 1 * 570a$

4*1a$\oplus 1 * 100a \oplus 1 * 16a \oplus 4 * 10a \oplus 1 * 16b \oplus 1 * 144a \oplus 1 * 144b$

1*16a$\oplus 1 * 10a \oplus 1 * 16b \oplus 4 * 44a \oplus 1 * 164a \oplus 1 * 570a \oplus 2 * 1046a \oplus 1 * 1184b \oplus 1 * 1184a$

7*1a$\oplus 2 * 100a \oplus 1 * 16a \oplus 4 * 10a \oplus 1 * 16b \oplus 1 * 144a \oplus 1 * 144b \oplus 1 * 1046a$

4*100a$\oplus 1 * 16a \oplus 1 * 10a \oplus 1 * 16b \oplus 1 * 44a \oplus 3 * 164a \oplus 3 * 416a \oplus 2 * 570a$

10*1a$\oplus 3 * 100a \oplus 1 * 16a \oplus 5 * 10a \oplus 1 * 16b$

2*16a$\oplus 2 * 10a \oplus 2 * 16b \oplus 5 * 44a \oplus 2 * 164a \oplus 1 * 416a \oplus 2 * 570a \oplus 2 * 1046a \oplus 2 * 1184b \oplus 2 * 1184a$

8*1a$\oplus 2 * 100a \oplus 1 * 16a \oplus 5 * 10a \oplus 1 * 16b \oplus 2 * 144a \oplus 2 * 144b \oplus 1 * 1046a$

3*100a$\oplus 1 * 16a \oplus 1 * 10a \oplus 1 * 16b \oplus 3 * 44a \oplus 2 * 164a \oplus 3 * 416a \oplus 1 * 570a \oplus 1 * 1046a$

9*1a$\oplus 3 * 100a \oplus 1 * 16a \oplus 4 * 10a \oplus 1 * 16b$

1*100a$\oplus 2 * 16a \oplus 2 * 10a \oplus 2 * 16b \oplus 2 * 44a \oplus 2 * 164a \oplus 2 * 416a \oplus 2 * 570a \oplus 1 * 1046a \oplus 1 * 1184b \oplus 1 * 1184a$

5*1a$\oplus 1 * 100a \oplus 1 * 16a \oplus 3 * 10a \oplus 1 * 16b \oplus 1 * 144a \oplus 1 * 144b \oplus 1 * 1046a$

2*100a$\oplus 1 * 16a \oplus 1 * 10a \oplus 1 * 16b \oplus 2 * 44a \oplus 1 * 164a \oplus 1 * 416a \oplus 1 * 570a$

3*1a$\oplus 1 * 100a \oplus 2 * 10a \oplus 1 * 144a \oplus 1 * 144b \oplus 1 * 1046a$

1*16a$\oplus 1 * 10a \oplus 1 * 16b \oplus 1 * 44a \oplus 1 * 164a \oplus 1 * 416a \oplus 1 * 570a \oplus 1 * 1184b \oplus 1 * 1184a$

4*1a$\oplus 2 * 100a$

1*16a$\oplus 1 * 10a \oplus 1 * 16b \oplus 1 * 164a \oplus 1 * 416a \oplus 1 * 570a$

1*1a$\oplus 1 * 144a \oplus 1 * 144b \oplus 1 * 1046a$

1*44a

1*1a

1*416a

Socle length 25

FIGURE 5.25. Socle series of $P(44a)$

1*44a

1*1a$\oplus 1 * 10a$

1*100a$\oplus 1 * 44a \oplus 1 * 164a \oplus 1 * 416a \oplus 1 * 570a$

4*1a$\oplus 1 * 100a \oplus 3 * 10a \oplus 1 * 144b \oplus 1 * 144a$

1*100a$\oplus 1 * 10a \oplus 1 * 16b \oplus 1 * 16a \oplus 4 * 44a \oplus 1 * 164a \oplus 1 * 570a \oplus 1 * 1046a \oplus 1 * 1184b \oplus 1 * 1184a$

8*1a$\oplus 2 * 100a \oplus 5 * 10a \oplus 1 * 16b \oplus 1 * 16a \oplus 1 * 144b \oplus 1 * 144a \oplus 1 * 1046a$

4*100a$\oplus 2 * 10a \oplus 2 * 16b \oplus 2 * 16a \oplus 3 * 44a \oplus 4 * 164a \oplus 4 * 416a \oplus 3 * 570a$

13*1a$\oplus 4 * 100a \oplus 7 * 10a \oplus 1 * 16b \oplus 1 * 16a \oplus 1 * 144b \oplus 1 * 144a$

1*100a$\oplus 3 * 10a \oplus 3 * 16b \oplus 3 * 16a \oplus 7 * 44a \oplus 3 * 164a \oplus 1 * 416a \oplus 3 * 570a \oplus 3 * 1046a \oplus 2 * 1184b \oplus 2 * 1184a$

14*1a$\oplus 4 * 100a \oplus 8 * 10a \oplus 2 * 16b \oplus 2 * 16a \oplus 3 * 144b \oplus 3 * 144a \oplus 2 * 1046a$

6*100a$\oplus 2 * 10a \oplus 2 * 16b \oplus 2 * 16a \oplus 4 * 44a \oplus 5 * 164a \oplus 5 * 416a \oplus 3 * 570a \oplus 1 * 1046a$

18*1a$\oplus 6 * 100a \oplus 8 * 10a \oplus 2 * 16b \oplus 2 * 16a \oplus 1 * 144b \oplus 1 * 144a$

1*100a$\oplus 4 * 10a \oplus 4 * 16b \oplus 4 * 16a \oplus 7 * 44a \oplus 4 * 164a \oplus 3 * 416a \oplus 4 * 570a \oplus 2 * 1046a \oplus 2 * 1184b \oplus 2 * 1184a$

12*1a$\oplus 3 * 100a \oplus 6 * 10a \oplus 1 * 16b \oplus 1 * 16a \oplus 2 * 144b \oplus 2 * 144a \oplus 2 * 1046a$

3*100a$\oplus 2 * 10a \oplus 2 * 16b \oplus 2 * 16a \oplus 5 * 44a \oplus 3 * 164a \oplus 2 * 416a \oplus 2 * 570a \oplus 1 * 1046a$

10*1a$\oplus 4 * 100a \oplus 4 * 10a \oplus 1 * 16b \oplus 1 * 16a \oplus 1 * 144b \oplus 1 * 144a \oplus 1 * 1046a$

1*100a$\oplus 2 * 10a \oplus 1 * 16b \oplus 1 * 16a \oplus 3 * 44a \oplus 3 * 164a \oplus 2 * 416a \oplus 3 * 570a \oplus 1 * 1184b \oplus 1 * 1184a$

8*1a$\oplus 3 * 100a \oplus 2 * 10a \oplus 1 * 144b \oplus 1 * 144a$

2*10a$\oplus 2 * 16b \oplus 2 * 16a \oplus 2 * 44a \oplus 2 * 164a \oplus 1 * 416a \oplus 2 * 570a$

3*1a$\oplus 1 * 100a \oplus 1 * 144b \oplus 1 * 144a \oplus 2 * 1046a$

1*10a$\oplus 1 * 16b \oplus 1 * 16a \oplus 2 * 44a \oplus 1 * 164a \oplus 1 * 570a$

4*1a$\oplus 2 * 100a$

1*10a$\oplus 1 * 164a \oplus 1 * 416a \oplus 1 * 570a$

1*1a

1*44a

Socle length 25

FIGURE 5.26. Socle series of $P(16a)$

$1*16a$
$1*100a \oplus 1 * 16b \oplus 1 * 144b \oplus 1 * 1046a \oplus 1 * 1184a$
$1*1a \oplus 3 * 16a \oplus 2 * 16b \oplus 3 * 10a \oplus 1 * 164a \oplus 1 * 416a$
$2*1a \oplus 5 * 100a \oplus 1 * 416a \oplus 1 * 144b \oplus 1 * 570a \oplus 1 * 1046a \oplus 1 * 1184a$
$2*1a \oplus 5 * 16a \oplus 2 * 16b \oplus 6 * 10a \oplus 1 * 44a \oplus 2 * 164a \oplus 1 * 144a \oplus 1 * 570a$
$6*1a \oplus 5 * 100a \oplus 1 * 10a \oplus 1 * 44a \oplus 1 * 416a \oplus 1 * 144b \oplus 1 * 144a \oplus 2 * 1046a \oplus 2 * 1184b$
$3*1a \oplus 4 * 16a \oplus 4 * 16b \oplus 7 * 10a \oplus 2 * 44a \oplus 3 * 164a \oplus 1 * 416a \oplus 1 * 570a$
$7*1a \oplus 8 * 100a \oplus 1 * 44a \oplus 1 * 164a \oplus 1 * 416a \oplus 1 * 144b \oplus 1 * 144a \oplus 1 * 570a \oplus 2 * 1046a \oplus 1 * 1184b$
$5*1a \oplus 1 * 100a \oplus 2 * 16a \oplus 6 * 16b \oplus 6 * 10a \oplus 3 * 44a \oplus 2 * 164a \oplus 1 * 416a \oplus 2 * 570a \oplus 1 * 1046a$
$8*1a \oplus 4 * 100a \oplus 2 * 16a \oplus 2 * 16b \oplus 3 * 10a \oplus 2 * 44a \oplus 2 * 164a \oplus 1 * 416a \oplus 2 * 144b \oplus 1 * 144a \oplus 1 * 570a \oplus 2 * 1046a \oplus 3 * 1184a$
$5*1a \oplus 2 * 100a \oplus 2 * 16a \oplus 4 * 16b \oplus 9 * 10a \oplus 2 * 44a \oplus 4 * 164a \oplus 2 * 416a \oplus 1 * 144b \oplus 1 * 144a \oplus 2 * 570a \oplus 1 * 1046a$
$9*1a \oplus 7 * 100a \oplus 1 * 16a \oplus 1 * 16b \oplus 1 * 10a \oplus 2 * 44a \oplus 1 * 164a \oplus 1 * 416a \oplus 1 * 144b \oplus 1 * 144a \oplus 1 * 570a \oplus 1 * 1046a \oplus 1 * 1184a$
$4*1a \oplus 2 * 100a \oplus 4 * 16a \oplus 1 * 16b \oplus 4 * 10a \oplus 4 * 44a \oplus 2 * 164a \oplus 1 * 416a \oplus 3 * 570a \oplus 1 * 1046a$
$8*1a \oplus 3 * 100a \oplus 2 * 16a \oplus 2 * 16b \oplus 3 * 10a \oplus 1 * 44a \oplus 1 * 164a \oplus 1 * 416a \oplus 1 * 144b \oplus 2 * 144a \oplus 2 * 1046a \oplus 2 * 1184b$
$2*1a \oplus 2 * 100a \oplus 4 * 16a \oplus 2 * 16b \oplus 6 * 10a \oplus 2 * 44a \oplus 3 * 164a \oplus 2 * 416a \oplus 3 * 570a$
$8*1a \oplus 6 * 100a \oplus 1 * 44a \oplus 1 * 416a \oplus 1 * 144b \oplus 1 * 144a \oplus 2 * 1046a \oplus 1 * 1184b$
$2*1a \oplus 4 * 16a \oplus 3 * 16b \oplus 7 * 10a \oplus 1 * 44a \oplus 3 * 164a \oplus 1 * 416a \oplus 1 * 570a$
$4*1a \oplus 3 * 100a \oplus 1 * 164a \oplus 1 * 144b \oplus 1 * 144a \oplus 1 * 570a \oplus 1 * 1046a \oplus 2 * 1184a$
$2*1a \oplus 1 * 100a \oplus 1 * 16b \oplus 2 * 44a \oplus 1 * 416a \oplus 1 * 570a \oplus 1 * 1046a$
$3*1a \oplus 2 * 16a \oplus 2 * 16b \oplus 2 * 10a \oplus 1 * 164a \oplus 1 * 144b$
$2*100a \oplus 1 * 44a \oplus 1 * 164a \oplus 1 * 416a \oplus 1 * 570a \oplus 1 * 1046a$
$3*1a \oplus 1 * 100a \oplus 1 * 16b \oplus 1 * 10a$
$1*16a \oplus 1 * 16b \oplus 1 * 10a \oplus 1 * 164a$
$1*100a \oplus 1 * 1046a$
$1*16a$

Socle Length 25

FIGURE 5.27. Socle series of $P(16b)$

$1*16b$
$1*100a \oplus 1 * 16a \oplus 1 * 144a \oplus 1 * 1046a \oplus 1 * 1184b$
$1*1a \oplus 3 * 16b \oplus 2 * 16a \oplus 3 * 10a \oplus 1 * 164a \oplus 1 * 416a$
$2*1a \oplus 5 * 100a \oplus 1 * 416a \oplus 1 * 144a \oplus 1 * 570a \oplus 1 * 1046a \oplus 1 * 1184b$
$2*1a \oplus 5 * 16b \oplus 2 * 16a \oplus 6 * 10a \oplus 1 * 44a \oplus 2 * 164a \oplus 1 * 144b \oplus 1 * 570a$
$6*1a \oplus 5 * 100a \oplus 1 * 10a \oplus 1 * 44a \oplus 1 * 416a \oplus 1 * 144a \oplus 1 * 144b \oplus 2 * 1046a \oplus 2 * 1184a$
$3*1a \oplus 4 * 16b \oplus 4 * 16a \oplus 7 * 10a \oplus 2 * 44a \oplus 3 * 164a \oplus 1 * 416a \oplus 1 * 570a$
$7*1a \oplus 8 * 100a \oplus 1 * 44a \oplus 1 * 164a \oplus 1 * 416a \oplus 1 * 144a \oplus 1 * 144b \oplus 1 * 570a \oplus 2 * 1046a \oplus 1 * 1184a$
$5*1a \oplus 1 * 100a \oplus 2 * 16b \oplus 6 * 16a \oplus 6 * 10a \oplus 3 * 44a \oplus 2 * 164a \oplus 1 * 416a \oplus 2 * 570a \oplus 1 * 1046a$
$8*1a \oplus 4 * 100a \oplus 2 * 16b \oplus 2 * 16a \oplus 3 * 10a \oplus 2 * 44a \oplus 2 * 164a \oplus 1 * 416a \oplus 2 * 144a \oplus 1 * 144b \oplus 1 * 570a \oplus 2 * 1046a \oplus 3 * 1184b$
$5*1a \oplus 2 * 100a \oplus 2 * 16b \oplus 4 * 16a \oplus 9 * 10a \oplus 2 * 44a \oplus 4 * 164a \oplus 2 * 416a \oplus 1 * 144a \oplus 1 * 144b \oplus 2 * 570a \oplus 1 * 1046a$
$9*1a \oplus 7 * 100a \oplus 1 * 16b \oplus 1 * 16a \oplus 1 * 10a \oplus 2 * 44a \oplus 1 * 164a \oplus 1 * 416a \oplus 1 * 144a \oplus 1 * 144b \oplus 1 * 570a \oplus 1 * 1046a \oplus 1 * 1184b$
$4*1a \oplus 2 * 100a \oplus 4 * 16b \oplus 1 * 16a \oplus 4 * 10a \oplus 4 * 44a \oplus 2 * 164a \oplus 1 * 416a \oplus 3 * 570a \oplus 1 * 1046a$
$8*1a \oplus 3 * 100a \oplus 2 * 16b \oplus 2 * 16a \oplus 3 * 10a \oplus 1 * 44a \oplus 1 * 164a \oplus 1 * 416a \oplus 1 * 144a \oplus 2 * 144b \oplus 2 * 1046a \oplus 2 * 1184a$
$2*1a \oplus 2 * 100a \oplus 4 * 16b \oplus 2 * 16a \oplus 6 * 10a \oplus 2 * 44a \oplus 3 * 164a \oplus 2 * 416a \oplus 3 * 570a$
$8*1a \oplus 6 * 100a \oplus 1 * 44a \oplus 1 * 416a \oplus 1 * 144a \oplus 1 * 144b \oplus 2 * 1046a \oplus 1 * 1184a$
$2*1a \oplus 4 * 16b \oplus 3 * 16a \oplus 7 * 10a \oplus 1 * 44a \oplus 3 * 164a \oplus 1 * 416a \oplus 1 * 570a$
$4*1a \oplus 3 * 100a \oplus 1 * 164a \oplus 1 * 144a \oplus 1 * 144b \oplus 1 * 570a \oplus 1 * 1046a \oplus 2 * 1184b$
$2*1a \oplus 1 * 100a \oplus 1 * 16a \oplus 2 * 44a \oplus 1 * 416a \oplus 1 * 570a \oplus 1 * 1046a$
$3*1a \oplus 2 * 16b \oplus 2 * 16a \oplus 2 * 10a \oplus 1 * 164a \oplus 1 * 144a$
$2*100a \oplus 1 * 44a \oplus 1 * 164a \oplus 1 * 416a \oplus 1 * 570a \oplus 1 * 1046a$
$3*1a \oplus 1 * 100a \oplus 1 * 16a \oplus 1 * 10a$
$1*16b \oplus 1 * 16a \oplus 1 * 10a \oplus 1 * 164a$
$1*100a \oplus 1 * 1046a$
$1*16b$

Socle length 25

FIGURE 5.28. Socle series of $P(10a)$

$1*10a$
$1*1a \oplus 2*100a \oplus 1*44a \oplus 1*1046a \oplus 1*1184a \oplus 1*1184b$
$2*1a \oplus 8*10a \oplus 3*16b \oplus 3*16a \oplus 1*164a \oplus 1*416a$
$3*1a \oplus 8*100a \oplus 3*44a \oplus 1*164a \oplus 2*416a \oplus 1*570a \oplus 2*1046a \oplus 2*1184a \oplus 2*1184b$
$7*1a \oplus 1*100a \oplus 16*10a \oplus 6*16b \oplus 6*16a \oplus 1*44a \oplus 2*164a \oplus 1*144b \oplus 1*144a \oplus 1*570a$
$6*1a \oplus 9*100a \oplus 2*10a \oplus 1*16b \oplus 1*16a \oplus 5*44a \oplus 2*164a \oplus 4*416a \oplus 1*144b \oplus 1*144a \oplus 1*570a \oplus 4*1046a \oplus 2*1184a \oplus 2*1184b$
$16*1a \oplus 2*100a \oplus 18*10a \oplus 7*16b \oplus 7*16a \oplus 2*44a \oplus 3*164a \oplus 1*416a \oplus 1*144b \oplus 1*144a \oplus 1*570a$
$6*1a \oplus 14*100a \oplus 1*10a \oplus 7*44a \oplus 4*164a \oplus 4*416a \oplus 1*144b \oplus 1*144a \oplus 3*570a \oplus 4*1046a \oplus 3*1184a \oplus 3*1184b$
$20*1a \oplus 2*100a \oplus 21*10a \oplus 6*16b \oplus 6*16a \oplus 3*44a \oplus 2*164a \oplus 1*416a \oplus 2*144b \oplus 2*144a \oplus 2*570a \oplus 1*1046a$
$8*1a \oplus 11*100a \oplus 4*10a \oplus 3*16b \oplus 3*16a \oplus 8*44a \oplus 6*164a \oplus 5*416a \oplus 1*144b \oplus 1*144a \oplus 4*570a \oplus 5*1046a \oplus 3*1184a \oplus 3*1184b$
$21*1a \oplus 3*100a \oplus 22*10a \oplus 9*16b \oplus 9*16a \oplus 2*44a \oplus 4*164a \oplus 2*416a \oplus 3*144b \oplus 3*144a \oplus 2*570a \oplus 1*1046a$
$9*1a \oplus 13*100a \oplus 2*10a \oplus 1*16b \oplus 1*16a \oplus 8*44a \oplus 4*164a \oplus 5*416a \oplus 1*144b \oplus 1*144a \oplus 4*570a \oplus 4*1046a \oplus 3*1184a \oplus 3*1184b$
$19*1a \oplus 3*100a \oplus 18*10a \oplus 4*16b \oplus 4*16a \oplus 4*44a \oplus 2*164a \oplus 1*416a \oplus 1*144b \oplus 1*144a \oplus 3*570a \oplus 1*1046a$
$8*1a \oplus 10*100a \oplus 3*10a \oplus 3*16b \oplus 3*16a \oplus 6*44a \oplus 4*164a \oplus 4*416a \oplus 2*144b \oplus 2*144a \oplus 2*570a \oplus 4*1046a \oplus 3*1184a \oplus 3*1184b$
$12*1a \oplus 2*100a \oplus 19*10a \oplus 6*16b \oplus 6*16a \oplus 2*44a \oplus 3*164a \oplus 2*416a \oplus 3*570a$
$8*1a \oplus 10*100a \oplus 4*44a \oplus 2*164a \oplus 3*416a \oplus 1*144b \oplus 1*144a \oplus 4*1046a \oplus 2*1184a \oplus 2*1184b$
$7*1a \oplus 14*10a \oplus 7*16b \oplus 7*16a \oplus 2*44a \oplus 3*164a \oplus 1*416a \oplus 2*570a$
$5*1a \oplus 8*100a \oplus 2*44a \oplus 1*164a \oplus 2*416a \oplus 1*144b \oplus 1*144a \oplus 1*570a \oplus 2*1046a \oplus 2*1184a \oplus 2*1184b$
$2*1a \oplus 1*100a \oplus 7*10a \oplus 2*44a \oplus 1*164a \oplus 1*416a \oplus 1*570a \oplus 1*1046a$
$4*1a \oplus 2*100a \oplus 2*10a \oplus 2*16b \oplus 2*16a \oplus 1*164a \oplus 1*1184a \oplus 1*1184b$
$2*100a \oplus 4*10a \oplus 1*44a \oplus 1*164a \oplus 1*416a \oplus 1*570a \oplus 1*1046a$
$3*1a \oplus 3*100a \oplus 1*16b \oplus 1*16a \oplus 1*1184a \oplus 1*1184b$
$5*10a \oplus 1*16b \oplus 1*16a \oplus 1*44a \oplus 1*164a$
$1*1a \oplus 2*100a \oplus 1*1046a \oplus 1*1184a \oplus 1*1184b$
$1*10a$

Socle length 25

FIGURE 5.29. Socle series of $P(164a)$

$1*164a$
$1*1a \oplus 1*100a$
$1*16b \oplus 1*10a \oplus 1*16a \oplus 1*44a \oplus 1*164a \oplus 1*570a$
$4*1a \oplus 2*100a \oplus 1*10a \oplus 1*144b \oplus 1*144a \oplus 1*1046a$
$1*100a \oplus 2*16b \oplus 2*10a \oplus 2*16a \oplus 1*44a \oplus 2*164a \oplus 1*416a \oplus 2*570a$
$8*1a \oplus 4*100a \oplus 2*10a \oplus 1*144b \oplus 1*144a \oplus 1*1046a$
$3*16b \oplus 3*10a \oplus 3*16a \oplus 4*44a \oplus 4*164a \oplus 1*416a \oplus 2*570a \oplus 1*1046a \oplus 1*1184b \oplus 1*1184a$
$11*1a \oplus 4*100a \oplus 1*16b \oplus 4*10a \oplus 1*16a \oplus 2*144b \oplus 2*144a \oplus 1*1046a$
$4*100a \oplus 2*16b \oplus 2*10a \oplus 2*16a \oplus 3*44a \oplus 4*164a \oplus 3*416a \oplus 3*570a \oplus 1*1046a$
$13*1a \oplus 5*100a \oplus 2*16b \oplus 6*10a \oplus 2*16a \oplus 1*144b \oplus 1*144a \oplus 1*1046a$
$1*100a \oplus 4*16b \oplus 4*10a \oplus 4*16a \oplus 5*44a \oplus 5*164a \oplus 2*416a \oplus 3*570a \oplus 1*1046a \oplus 1*1184b \oplus 1*1184a$
$13*1a \oplus 5*100a \oplus 1*16b \oplus 4*10a \oplus 1*16a \oplus 2*144b \oplus 2*144a \oplus 2*1046a$
$3*100a \oplus 2*16b \oplus 2*10a \oplus 2*16a \oplus 4*44a \oplus 5*164a \oplus 2*416a \oplus 4*570a \oplus 1*1046a$
$13*1a \oplus 4*100a \oplus 1*16b \oplus 4*10a \oplus 1*16a \oplus 2*144b \oplus 2*144a \oplus 1*1046a$
$1*100a \oplus 3*16b \oplus 3*10a \oplus 3*16a \oplus 3*44a \oplus 4*164a \oplus 2*416a \oplus 3*570a \oplus 1*1184b \oplus 1*1184a$
$9*1a \oplus 5*100a \oplus 2*10a \oplus 1*144b \oplus 1*144a \oplus 1*1046a$
$1*100a \oplus 3*16b \oplus 3*10a \oplus 3*16a \oplus 3*44a \oplus 3*164a \oplus 1*416a \oplus 2*570a \oplus 1*1046a$
$6*1a \oplus 2*100a \oplus 1*16b \oplus 1*10a \oplus 1*16a \oplus 1*144b \oplus 1*144a \oplus 1*1046a$
$1*100a \oplus 1*10a \oplus 2*44a \oplus 2*164a \oplus 1*416a \oplus 2*570a \oplus 1*1046a$
$6*1a \oplus 1*100a \oplus 1*16b \oplus 1*10a \oplus 1*16a \oplus 1*144b \oplus 1*144a$
$1*100a \oplus 1*16b \oplus 1*10a \oplus 1*16a \oplus 1*44a \oplus 1*164a \oplus 1*416a \oplus 1*570a$
$3*1a \oplus 1*100a \oplus 1*1046a$
$1*16b \oplus 1*10a \oplus 1*16a \oplus 1*44a \oplus 1*164a \oplus 1*570a$
$1*1a \oplus 1*100a$
$1*164a$

Socle length 25

FIGURE 5.30. Socle series of $P(100a)$

1*100a
1*16b⊕2 * 10a ⊕ 1 * 16a ⊕ 1 * 164a
1*1a⊕4 * 100a ⊕ 1 * 44a ⊕ 2 * 1046a ⊕ 1 * 1184a ⊕ 1 * 1184b
3*1a⊕5 * 16b ⊕ 8 * 10a ⊕ 5 * 16a ⊕ 1 * 44a ⊕ 2 * 164a ⊕ 1 * 570a
7*1a⊕9 * 100a ⊕ 1 * 10a ⊕ 1 * 44a ⊕ 1 * 164a ⊕ 3 * 416a ⊕ 1 * 144b ⊕ 1 * 144a ⊕ 1 * 1046a ⊕ 1 * 1184a ⊕ 1 * 1184b
4*1a⊕1 * 100a ⊕ 5 * 16b ⊕ 9 * 10a ⊕ 5 * 16a ⊕ 2 * 44a ⊕ 4 * 164a ⊕ 1 * 416a ⊕ 3 * 570a
9*1a⊕10 * 100a ⊕ 2 * 10a ⊕ 4 * 44a ⊕ 2 * 144b ⊕ 2 * 144a ⊕ 1 * 570a ⊕ 5 * 1046a ⊕ 1 * 1184a ⊕ 1 * 1184b
9*1a⊕8 * 16b ⊕ 14 * 10a ⊕ 8 * 16a ⊕ 4 * 44a ⊕ 4 * 164a ⊕ 2 * 416a ⊕ 1 * 144b ⊕ 1 * 144a ⊕ 2 * 570a ⊕ 1 * 1046a
13*1a⊕10 * 100a ⊕ 1 * 16b ⊕ 2 * 10a ⊕ 1 * 16a ⊕ 1 * 44a ⊕ 4 * 164a ⊕ 4 * 416a ⊕ 2 * 144b ⊕ 2 * 144a ⊕ 1 * 570a ⊕ 1 * 1046a ⊕ 2 * 1184a ⊕ 2 * 1184b
9*1a⊕3 * 100a ⊕ 4 * 16b ⊕ 11 * 10a ⊕ 4 * 16a ⊕ 4 * 44a ⊕ 5 * 164a ⊕ 3 * 416a ⊕ 4 * 570a ⊕ 1 * 1046a
15*1a⊕10 * 100a ⊕ 2 * 16b ⊕ 3 * 10a ⊕ 2 * 16a ⊕ 6 * 44a ⊕ 1 * 164a ⊕ 2 * 144b ⊕ 2 * 144a ⊕ 3 * 570a ⊕ 5 * 1046a ⊕ 1 * 1184a ⊕ 1 * 1184b
8*1a⊕2 * 100a ⊕ 7 * 16b ⊕ 13 * 10a ⊕ 7 * 16a ⊕ 6 * 44a ⊕ 5 * 164a ⊕ 2 * 416a ⊕ 2 * 144b ⊕ 2 * 144a ⊕ 3 * 570a ⊕ 2 * 1046a
13*1a⊕7 * 100a ⊕ 2 * 16b ⊕ 3 * 10a ⊕ 2 * 16a ⊕ 1 * 44a ⊕ 3 * 164a ⊕ 3 * 416a ⊕ 2 * 144b ⊕ 2 * 144a ⊕ 1 * 570a ⊕ 1 * 1046a ⊕ 2 * 1184a ⊕ 2 * 1184b
8*1a⊕3 * 100a ⊕ 3 * 16b ⊕ 10 * 10a ⊕ 3 * 16a ⊕ 3 * 44a ⊕ 4 * 164a ⊕ 3 * 416a ⊕ 4 * 570a ⊕ 1 * 1046a
13*1a⊕10 * 100a ⊕ 2 * 16b ⊕ 2 * 10a ⊕ 2 * 16a ⊕ 3 * 44a ⊕ 1 * 164a ⊕ 1 * 416a ⊕ 1 * 144b ⊕ 1 * 144a ⊕ 1 * 570a ⊕ 3 * 1046a ⊕ 1 * 1184a ⊕ 1 * 1184b
4*1a⊕2 * 100a ⊕ 6 * 16b ⊕ 10 * 10a ⊕ 6 * 16a ⊕ 4 * 44a ⊕ 5 * 164a ⊕ 1 * 416a ⊕ 4 * 570a
8*1a⊕6 * 100a ⊕ 1 * 44a ⊕ 1 * 164a ⊕ 2 * 416a ⊕ 2 * 144b ⊕ 2 * 144a ⊕ 3 * 1046a ⊕ 1 * 1184a ⊕ 1 * 1184b
3*1a⊕3 * 16b ⊕ 8 * 10a ⊕ 3 * 16a ⊕ 3 * 44a ⊕ 2 * 164a ⊕ 1 * 416a ⊕ 1 * 570a ⊕ 1 * 1046a
5*1a⊕4 * 100a ⊕ 1 * 16b ⊕ 1 * 10a ⊕ 1 * 16a ⊕ 1 * 164a ⊕ 1 * 144b ⊕ 1 * 144a ⊕ 1 * 570a ⊕ 1 * 1184a ⊕ 1 * 1184b
2*1a⊕2 * 100a ⊕ 2 * 10a ⊕ 1 * 44a ⊕ 1 * 164a ⊕ 2 * 416a ⊕ 1 * 570a ⊕ 1 * 1046a
5*1a⊕3 * 100a ⊕ 2 * 16b ⊕ 2 * 10a ⊕ 2 * 16a ⊕ 1 * 164a
1*100a⊕1 * 16b ⊕ 3 * 10a ⊕ 1 * 16a ⊕ 2 * 44a ⊕ 1 * 164a ⊕ 1 * 570a ⊕ 1 * 1046a
1*1a⊕1 * 100a ⊕ 1 * 1046a ⊕ 1 * 1184a ⊕ 1 * 1184b
1*16b⊕2 * 10a ⊕ 1 * 16a ⊕ 1 * 164a
1*100a

Socle length 25

FIGURE 5.31. Socle series of $P(1a)$

1*1a
1*10a⊕1 * 44a ⊕ 1 * 164a ⊕ 1 * 416a ⊕ 2 * 570a ⊕ 1 * 1046a
9*1a⊕1 * 100a ⊕ 1 * 16a ⊕ 2 * 10a ⊕ 1 * 16b ⊕ 2 * 144b ⊕ 2 * 144a ⊕ 1 * 1046a ⊕ 1 * 1184b ⊕ 1 * 1184a
1*1a⊕3 * 100a ⊕ 2 * 16a ⊕ 3 * 10a ⊕ 2 * 16b ⊕ 4 * 44a ⊕ 4 * 164a ⊕ 3 * 416a ⊕ 3 * 570a ⊕ 1 * 1046a
18*1a⊕7 * 100a ⊕ 2 * 16a ⊕ 7 * 10a ⊕ 2 * 16b ⊕ 2 * 144b ⊕ 2 * 144a ⊕ 1 * 1046a
4*100a⊕6 * 16a ⊕ 6 * 10a ⊕ 6 * 16b ⊕ 8 * 44a ⊕ 8 * 164a ⊕ 4 * 416a ⊕ 7 * 570a ⊕ 3 * 1046a ⊕ 2 * 1184b ⊕ 2 * 1184a
28*1a⊕9 * 100a ⊕ 3 * 16a ⊕ 16 * 10a ⊕ 3 * 16b ⊕ 5 * 144b ⊕ 5 * 144a ⊕ 3 * 1046a
1*1a⊕9 * 100a ⊕ 7 * 16a ⊕ 6 * 10a ⊕ 7 * 16b ⊕ 13 * 44a ⊕ 11 * 164a ⊕ 7 * 416a ⊕ 7 * 570a ⊕ 4 * 1046a ⊕ 2 * 1184b ⊕ 2 * 1184a
41*1a⊕13 * 100a ⊕ 5 * 16a ⊕ 20 * 10a ⊕ 5 * 16b ⊕ 4 * 144b ⊕ 4 * 144a ⊕ 2 * 1046a
9*100a⊕8 * 16a ⊕ 8 * 10a ⊕ 8 * 16b ⊕ 14 * 44a ⊕ 13 * 164a ⊕ 10 * 416a ⊕ 10 * 570a ⊕ 4 * 1046a ⊕ 3 * 1184b ⊕ 3 * 1184a
42*1a⊕15 * 100a ⊕ 5 * 16a ⊕ 21 * 10a ⊕ 5 * 16b ⊕ 6 * 144b ⊕ 6 * 144a ⊕ 4 * 1046a
8*100a⊕9 * 16a ⊕ 9 * 10a ⊕ 9 * 16b ⊕ 18 * 44a ⊕ 13 * 164a ⊕ 8 * 416a ⊕ 9 * 570a ⊕ 5 * 1046a ⊕ 3 * 1184b ⊕ 3 * 1184a
42*1a⊕13 * 100a ⊕ 4 * 16a ⊕ 19 * 10a ⊕ 4 * 16b ⊕ 5 * 144b ⊕ 5 * 144a ⊕ 4 * 1046a
8*100a⊕8 * 16a ⊕ 8 * 10a ⊕ 8 * 16b ⊕ 12 * 44a ⊕ 13 * 164a ⊕ 9 * 416a ⊕ 11 * 570a ⊕ 2 * 1046a ⊕ 2 * 1184b ⊕ 2 * 1184a
37*1a⊕13 * 100a ⊕ 2 * 16a ⊕ 12 * 10a ⊕ 2 * 16b ⊕ 4 * 144b ⊕ 4 * 144a ⊕ 3 * 1046a
4*100a⊕8 * 16a ⊕ 8 * 10a ⊕ 8 * 16b ⊕ 10 * 44a ⊕ 9 * 164a ⊕ 5 * 416a ⊕ 7 * 570a ⊕ 2 * 1046a ⊕ 2 * 1184b ⊕ 2 * 1184a
21*1a⊕8 * 100a ⊕ 2 * 16a ⊕ 7 * 10a ⊕ 2 * 16b ⊕ 4 * 144b ⊕ 4 * 144a ⊕ 4 * 1046a
3*100a⊕4 * 16a ⊕ 5 * 10a ⊕ 4 * 16b ⊕ 8 * 44a ⊕ 6 * 164a ⊕ 3 * 416a ⊕ 6 * 570a ⊕ 2 * 1046a ⊕ 1 * 1184b ⊕ 1 * 1184a
18*1a⊕5 * 100a ⊕ 2 * 16a ⊕ 2 * 10a ⊕ 2 * 16b ⊕ 2 * 144b ⊕ 2 * 144a ⊕ 2 * 1046a
2*100a⊕3 * 16a ⊕ 4 * 10a ⊕ 3 * 16b ⊕ 3 * 44a ⊕ 6 * 164a ⊕ 4 * 416a ⊕ 6 * 570a
14*1a⊕5 * 100a ⊕ 2 * 144b ⊕ 2 * 144a ⊕ 2 * 1046a
3*16a⊕3 * 10a ⊕ 3 * 16b ⊕ 4 * 44a ⊕ 3 * 164a ⊕ 1 * 416a ⊕ 2 * 570a
6*1a⊕1 * 100a ⊕ 2 * 1046a
1*10a⊕1 * 44a ⊕ 1 * 164a ⊕ 1 * 416a ⊕ 2 * 570a
1*1a

Socle length 25

Table 5.8. Correspondence between the composition factors of the projective indecomposable $e_H\mathbb{F}_2A_{12}e_H$-modules and the composition factors of the projective indecomposable $\mathbb{F}_2A_{12}$-modules

Simples of A_{12} in the principal block			1a	10a	16a	16b	44a	100a	144a	144b	164a	416a	570a	1046a	1184a	1184b
P.I.M.s A_{12}	Comp. name	Dir. name														
P(1a)	sfnpim1a	pim1a	1a	4b	4a	4c	6a	2a	16c	16b	8a	16a	18a	32a	52a	52b
P(10a)	sfpim4a	pim4a	1a	4a	4c	4b	6a	2a	16c	16b	8a	16a	18a	32a	52a	52b
P(16a)	pim4c	pim4c	1a	4c	4a	4b	6a	2a	16c	16b	8a	16a	18a	32a	52a	52b
P(16b)	pim4b	pim4b	1a	4c	4b	4a	6a	2a	16b	16c	8a	16a	18a	32a	52b	52a
P(44a)	sfpim6a	pim6a	1a	4a	4c	4b	6a	2a	16c	16b	8a	16a	18a	32a	52b	52a
P(100a)	sfpim2a	pim2a	1a	4b	4c	4a	6a	2a	16c	16b	8a	16a	18a	32a	52a	52b
P(144a)	pim16b	pim16b	1a	4c	4b	4a	6a	2a	16a	16c	8a	16b	18a	32a	52b	52a
P(144b)	pim16c	pim16c	1a	4c	4a	4b	6a	2a	16c	16a	8a	16b	18a	32a	52a	52b
P(164a)	sfpim8a	pim8a	1a	4b	4c	4a	6a	2a	16c	16b	8a	16a	18a	32a	52b	52a
P(416a)	pim16a	sfpim6a	1a	4b	4a	4c	6a	2a	16b	16c	8a	16a	18a	32a	52b	52a
P(570a)	sfnpim18a	pim18a	1a	4b	4a	4c	6a	2a	16c	16a	8a	16b	18a	32a	52a	52b
P(1046a)	sfpim32a	pim32a	1a	4b	4a	4c	6a	2a	16c	16b	8a	16a	18a	32a	52a	52b
P(1184a)	pim52a	pim52a	1a	4a	4b	4c	6a	2a	16c	16b	8a	16a	18a	32a	52a	52b
P(1184b)	pim52b	pim52b	1a	4a	4c	4b	6a	2a	16b	16c	8a	16a	18a	32a	52b	52a

Appendix A

The GAP procedure MyOrbitIntersectionMatrix

```
MyOrbitIntersectionMatrix:=function(sub,x)
local i,j,k,len,o,v,y,a,b,p;
v:=[];
len:=sub.nrsuborbits;
o:=sub.o;
     for i in [1..len] do
     a:=[];
     b:=[];
               for j in sub.suborbs[i] do
               y:=o!.op(o[j],x);
               k:=sub!.suborbnr[Position(o,y)];
                         if k in b then
                         p:=Position(b,k);
                         a[p]:=a[p]+1;
                         else
                         Add(b,k);
                         Add(a,1);
                         fi;
               od;
     v[i]:=[a*Z(2)^0,b];
     od;
return v;
end;
```

Appendix B

The GAP procedure MyVectortimesSparseMat

```
MyVectorTimesSparseMat:=function(w,m)
local v,i,j,a,b,k;
v:=CVEC_ZeroMat(1,Length(m),2,1);
v:=v[1];
     if Length(m)<>Length(w) then
     Print("dim not correct");
     return;
     fi;
               for i in [1..Length(w)] do
               a:=m[i][1]*Z(2)^0;
               b:=m[i][2];
               k:=w[i]*a;
                         for j in [1..Length(a)] do
                         v[b[j]]:=v[b[j]]+k[j];
                         od;
               od;
     return v;
     end;
```

Appendix C

The GAP procedure VectorTimesWord

```
VectorTimesWord:=function(vec,word)
local list,i,j,w,sum;
list:=[];
     for i in [1..Length(word)] do
     w:=ShallowCopy(vec);
             for j in [1..Length(word[i])] do
             w:=MyVectorTimesSparseMat(w,word[i][j]);
             od;
     Add(list,w);
     w:=ShallowCopy(vec);
     od;
sum:=Sum(list);
return sum;
end;
```

Appendix D

The GAP procedure MySpinningwwords

```
MySpinningwwords:=function(vec,word)
local coefflist,basis,z,m,t,vectimesword,newl,newz,coef;
coefflist:=[];
m:=RandomMat(1,2,GF(2));
m:=CMat(m);
basis:=EmptySemiEchelonBasis(m);
z:=ZeroVector(10000,vec);
vectimesword:=ShallowCopy(vec);
t:=ShallowCopy(vec);
     while CleanRow(basis,vec,true,z)=false do
     vectimesword:=VectorTimesWord(vectimesword,word);
     vec:=ShallowCopy(vectimesword);
     Add(coefflist,ShallowCopy(z));
     od;
coefflist:=CMat(coefflist);
newl:=ExtractSubMatrix(coefflist,[1..Length(coefflist)],[1..Length(coefflist)]);
newz:=ELMS_LIST(z,[1..Length(coefflist)]);
coef:=newz*newl^-1;
return coef;
end;
```

Appendix E

The GAP procedure MyVecttimesPolyinMatRingwwords

```
MyVecttimesPolyinMatRingwwords:=function(vec,coeff,word)
local apower,sum,i;
apower:=vec;
sum:=[];
    for i in [2..Length(coeff)] do
    apower:=VectorTimesWord(apower,word);
            if coeff[i]<>0*Z(2) then
            sum:=sum+apower;
            fi;
    od;
                    if coeff[1]<>0*Z(2) then
                    sum:=sum+vec;
                    fi;
sum:=sum+VectorTimesWord(apower,word);
return sum;
end;
```

Appendix F

The GAP procedure MySpinningwgen

```
MySpinningwgen:=function(vec,gen,maxdim)
local l,len,cond,m,basis,lastvec,pos,z,j,y,i;
len:=Length(gen);
l:=[];
    for i in [1..len] do
    cond:=CVEC_ZeroMat(maxdim,maxdim,2,1);
    Add(l,cond);
    od;
            m:=RandomMat(1,2,GF(2));
            m:=CMat(m);
            basis:=EmptySemiEchelonBasis(m);
            z:=ZeroVector(maxdim,vec);
            lastvec:=1;
            pos:=1;
            CleanRow(basis,vec,true,z);
                while pos <= lastvec do
                    for j in [1..len] do
                    y:=MyVectorTimesSparseMat(basis[pos],gen[j]);
                        if CleanRow(basis,y,true,z)=false then
                        lastvec:=lastvec+1;
                        fi;
                    l[j][pos]:=ShallowCopy(z);
                    Print("j ",j,"");
```

```
          od;
        Print(Length(basis)," ");
        pos:=pos+1;
        od;
return l;
end;
```

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