Separation of Variables, Solving DEs Symbolically
Math 112, Fall 2009

First, we look at a familiar example, the differential equation

$$\frac{dy}{dx} = -\frac{x}{y},$$

whose solution curves are the circles

$$x^2 + y^2 = C.$$

We can check that these circles are solutions by differentiation; the question now is how they were obtained. The method of *separation of variables* works by putting all the xs on one side of the equation and all the ys on the other, giving

$$y\,dy = -x\,dx.$$

We then integrate each side separately:

$$\int y\,dy = -\int x\,dx,$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + k.$$

This gives the circles we were expecting:

$$x^2 + y^2 = C \qquad \text{where } C = 2k.$$

Example: Consider the population of a town. If there is no immigration or emigration, the rate at which the population is changing is often proportional to the population. In other words, the larger the population, the faster it is growing, because there are more people to have babies. Population growth or decay can be modelled by a differential equation that occurs frequently in practice.

$$\frac{dy}{dx} = ky.$$

Solve this differential equation using the method of separating variables.

Example: Find the solution to the differential equations, subject to the given initial conditions.

1. $2\frac{du}{dt} = u^2$, $u(0) = 1$.

2. $\frac{dz}{dy} = zy$, $z = 1$ when $y = 0$.

3. $\frac{dy}{dt} = 0.5(y - 200)$, $y = 50$ when $t = 0$.

Newton's Law of Heating and Cooling

Newton proposed that the temperature of a hot object decreases at a rate proportional to the difference between its temperature and that of its surroundings. Similarly, a cold object heats up at a rate proportional to the temperature difference between the object and its surroundings.

For example, a hot cup of coffee standing on the kitchen table cools at a rate proportional to the temperature difference between the coffee and the surrounding air. As the coffee cools, the rate at which it cools decreases, because the temperature difference between the coffee and the air decreases. In the long run, the rate of cooling tends to zero, and the temperature of the coffee approaches room temperature.

Let H be the temperature of the cup of coffee and let S be the surrounding temperature. Then

$$\frac{dH}{dt} = \alpha(H - S)$$

Example: When a murder is committed, the body, originally at 37°C, cools according to Newton's Law of Cooling. Suppose that after two hours the temperature is 35°C, and that the temperature of the surrounding air is constant at 20°C.

1. Find the temperature H, of the body as a function of t, the time in hours since the murder was committed.

2. Sketch a graph of temperature against time.

3. What happens to the temperature in the long run? Show this on the graph and algebraically.

4. If the body is found at a temperature of 30°C, when was the murder committed?

Example: As you know, when a course ends, students start to forget the material they have learned. One model (called the Ebbinghaus model) assumes that the rate at which a student remembers material is proportional to the difference between the material currently remembered and some positive constant, a.

1. Let $y = f(t)$ be the fraction of the original material remembered t weeks after the course has ended. Set up a differential equation for y. Your equation will contain two constants; the constant a is less than y for all t.

2. Solve the differential equation.

3. Describe the practical meaning (in terms of the amount remembered) of the constants in the solution $y = f(t)$.

Example: Kenyon College has 1600 students. On day 0, Sam, Kris, Judy, Selin, and 11 of their friends start a rumor that spreads logistically. A day later, 50 students know the rumor. Let $P(t)$ denote the number of people who knows the rumor. Given that $P(t)$ satisfies the following model

$$\frac{dP}{dT} = kP(1600 - P), P(0) = 15, \text{ and } P(1) = 50,$$

answer the following questions:

1. What happens over the next 10 days?

2. When is the rumor spreading the fastest?