Solutions to Review 3

1 a) $\frac{dy}{dt} = y^2 \sin t \qquad y(0) = \frac{1}{2}$

$$\int \frac{dy}{y^2} = \int \sin t \, dt$$

$$-\frac{1}{y} = -\cos t + C$$

$$y = \frac{1}{\cos t + C} \qquad \frac{1}{2} = \frac{1}{\cos 0 + C}$$

$$\frac{1}{2} = \frac{1}{1+C} \qquad \boxed{C = 1}$$

$$\boxed{y = \frac{1}{\cos t + 1}}$$

b) $\frac{dy}{dt} = yt \qquad \frac{dy}{y} = t\,dt \qquad y(0) = \frac{1}{2}$

$$\ln y = \frac{t^2}{2} + C$$

$$y = Ce^{\frac{t^2}{2}}$$

$$\boxed{\frac{1}{2} = C}$$

$$\boxed{y = \frac{1}{2} e^{\frac{t^2}{2}}}$$

c) $\frac{dx}{\tan x} = \frac{1+2\ln t}{t} \, dt$

$$\int \frac{\cos x}{\sin x} dx = \int \frac{1+2\ln t}{t} dt \qquad 1+2\ln t = w$$

$$\frac{2}{t} dt = dw$$

$\sin x = u$

$\cos x \, dx = du$

$$\int \frac{du}{u} = \frac{1}{2} \int w \, dw$$

$$\ln u = \frac{1}{2} \frac{w^2}{2} + C$$

$$u = Ce^{w^2/4}$$

$$\boxed{\sin x = Ce^{(1+2\ln t)^2/4}}$$

(d) $\displaystyle\int \frac{du}{u^2} = \int \frac{dx}{x(x+1)}$ $\qquad \displaystyle\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$

$1 = A(x+1) + Bx$

$\displaystyle -\frac{1}{u} = \int \frac{dx}{x} - \int \frac{dx}{x+1}$ $\qquad x=0 \quad \boxed{A=1}$

$x=-1 \quad \boxed{B=-1}$

$-\frac{1}{u} = \ln|x| - \ln|x+1| + C$

$u(1) = 1 \Rightarrow -1 = \ln \overset{0}{1} - \ln 2 + C$

$\boxed{C = \ln 2 - 1}$

$\boxed{-\frac{1}{u} = \ln|x| - \ln|x+1| + \ln 2 - 1}$

(e) $\displaystyle\int \frac{dy}{y\ln\left(\frac{y}{2}\right)} = -\int dt$

$\ln\left(\frac{y}{2}\right) = u \quad \Rightarrow \quad \displaystyle\int \frac{du}{u} = -t + c$

$\displaystyle\frac{\frac{1}{2}}{\frac{y}{2}}\,dy = du \qquad \ln u = -t + c \qquad \ln\left(\ln\frac{y}{2}\right) = -t + C \qquad y(0) = 1$

$\frac{1}{y}\,dy = du \qquad \ln\left(\ln\frac{1}{2}\right) = C$

$\boxed{\ln\left(\ln\frac{y}{2}\right) = -t + \ln\left(\ln\frac{1}{2}\right)}$

... it is not necessary.

(2) $\displaystyle\frac{dR}{dS} = k\,\frac{1}{S}$

$\displaystyle\int dR = \int \frac{k\,dS}{S}$

$\boxed{R = k\ln|S| + C}$

(3) $V = a^3 \qquad a = V^{1/3}$ $\qquad$ [surface area written in terms of volume]

$S = 6a^2 \qquad \boxed{S = 6V^{2/3}}$

(a) $\boxed{\displaystyle\frac{dV}{dt} = 6kV^{2/3}}$ $\qquad \displaystyle\int \frac{dV}{V^{2/3}} = \int 6k\,dt$

with $k < 0$.

(b) $\int \frac{dV}{V^{2/3}} = \int 6k\,dt$

$3V^{1/3} = 6kt + C$

$V^{1/3} = 2kt + C$

$\boxed{V = (2kt + C)^3}$

$t = 0 \quad V = V_0$

$V_0 = C^3 \qquad C = V_0^{1/3}$

$V = (2kt + V_0^{1/3})^3$

(c) Ice cube dissappears when $V = 0$

$(2kt + V_0^{1/3})^3 = 0$

$2kt + V_0^{1/3} = 0$

$\boxed{t = -\frac{V_0^{1/3}}{2k}}$

(4) (a) $\frac{dP}{dt} = k(800 - P)$

$\int \frac{dP}{800-P} = \int k\,dt$

$-\ln|800 - P| = kt + c$

$\ln|800 - P| = -kt + C$

$800 - P = Ce^{-kt}$

$P(0) = 500 \quad 800 - 500 = C \qquad \boxed{C = 300}$

$\boxed{P = 800 - 300e^{-kt}}$

(b) $P(2) = 700$

$700 = 800 - 300e^{-2k}$

$-100 = -300e^{-2k}$

$\frac{1}{3} = e^{-2k}$

$\ln\frac{1}{3} = -2k$

$\boxed{k = \frac{\ln 1/3}{-2}}$

(c) $\boxed{\text{As } t \to \infty \quad P \to 800}$

5 (a) $\sin x \approx x - \frac{1}{6}x^3 + \frac{1}{120}x^5$ (First I write the 5^{th} degree Taylor poly appr. at $x=0$ and then multiply it by x).

Hence

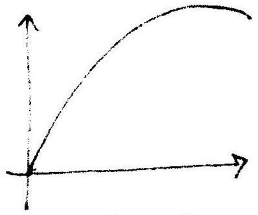
$x\sin x \approx x^2 - \frac{1}{6}x^4 + \frac{1}{120}x^6$

(b) First we have to find K_7 such that

$|f^7(x)| \leqslant K_7$

In maple graph the seventh derivative of $x\sin x$ on the interval $[0, \pi/2]$

plot $(|-7\sin x - x\cos x|, x = 0..\frac{\pi}{2})$

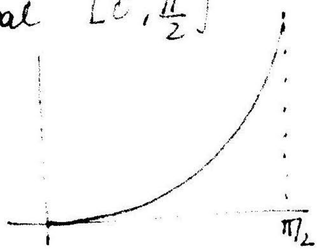


We can take $K_7 = 8$

Then by Taylor's theorem

$|f(x) - P_6(x)| \leqslant \frac{8\,|x|^7}{7!}$

To see the behaviour of the bound $\frac{8|x|^7}{7!}$ we plot this on the interval $[0, \frac{\pi}{2}]$



We can see that $\frac{8|x|^7}{7!}$ is the largest at $\pi/2$.
So we evaluate $\frac{8|x|^7}{7!}$ at $\pi/2$

$= 0.037455$.

So the largest value for the error bound is 0.037455.

So we know that $P_6(x)$ will commit an error no more than 0.037455 for all $x \in [0, \pi/2]$.

(c) To find the actual max. error committed by $P_6(x)$ we have to compute

$|f(\pi/2) - P_6(\pi/2)| = \boxed{0.0071}$

(6) $4\arctan x \approx 4\left(x - \frac{1}{3}x^3\right)$ → Taylor poly opp. of degree 3 to $\arctan x$ 3

$4\arctan x = 4x - \frac{4}{3}x^3$

Now plug in 1 for x

$4\arctan 1 \approx 4 - \frac{4}{3}$

$4\frac{\pi}{4} \approx \frac{8}{3}$

$\pi \approx \frac{8}{3} = 2.6667$ (which you can see is not a good approximation)

(7) (a)

(b) $\int_5^\infty \frac{dx}{(x-1)^{3/2}} = \lim_{t\to\infty} \int_5^t \frac{dx}{(x-1)^{3/2}}$ $\quad x-1=u$, $dx = du$ $\quad \int \frac{du}{u^{3/2}} = -2u^{-1/2}$

$= \lim_{t\to\infty} -2(x-1)^{-1/2}\Big|_5^t$

$= \lim_{t\to\infty} \cancel{-2(t-1)^{-1/2}}^{\,0} + 2(4)^{-1/2}$

$= \frac{2}{\sqrt{4}} = \boxed{1}$

(c) $\int_0^\infty \frac{x^2\,dx}{4x^3+5} = \lim_{t\to\infty} \int_0^t \frac{x^2\,dx}{4x^3+5}$ $\quad 4x^3+5 = u$, $12x^2\,dx = du$ $\quad \frac{1}{12}\int \frac{du}{u} = \frac{1}{12}\ln|u|$

$= \lim_{t\to\infty} \frac{1}{12}\ln|4x^3+5|\Big|_0^t$

$= \lim_{t\to\infty} \underbrace{\frac{1}{12}\ln(4t^3+5)}_{\to\infty} - \frac{1}{12}\ln 5$

$= \infty$ Hence the integral is divergent.

(d) $\displaystyle\int_0^6 \frac{dx}{(x-4)^{2/3}}$ this integral is improper since the integrand is unbounded at $x=4$

$$\int_0^4 \frac{dx}{(x-4)^{2/3}} + \int_4^6 \frac{dx}{(x-4)^{2/3}} = \lim_{t\to 4^-} \int_0^t \frac{dx}{(x-4)^{2/3}} + \lim_{t\to 4^+} \int_t^6 \frac{dx}{(x-4)^{2/3}}$$

Let $x-4=u$, $dx=du$ $\qquad \int u^{-2/3}\,du = 3u^{1/3}$

$$= \lim_{t\to 4^-} 3(x-4)^{1/3}\Big|_0^t + \lim_{t\to 4^+} 3(x-4)^{1/3}\Big|_t^6$$

$$= \lim_{t\to 4^-} \cancel{3(t-4)^{1/3}} - 3(-4)^{1/3} + \lim_{t\to 4^+} 3(2)^{1/3} - 3(t-4)^{1/3}$$

$$= \boxed{-3(-4)^{1/3} + 3\,2^{1/3}}$$

(8) (a) compare with $\displaystyle\int_1^\infty \frac{dx}{x^3}$.

$\dfrac{1}{x^3+1} \leq \dfrac{1}{x^3}$ for all $x \in [1,\infty)$

Since $\displaystyle\int_1^\infty \frac{dx}{x^3}$ is convergent (You have to know why!)

therefore $\displaystyle\int_1^\infty \frac{dx}{x^3+1}$ is convergent.

(b) $\displaystyle\int_0^1 \frac{dx}{x^{19/20}}$ $\quad \displaystyle\lim_{t\to 0^+} \int_t^1 x^{-19/20}\,dx = \lim_{t\to 0^+} 20x^{1/20}\Big|_t^1$

$$= \lim_{t\to 0^+} 20 - 20t^{1/20}$$

$= \boxed{20}$

We did not need comparison here!

(c) $\int_1^\infty \frac{du}{u+u^2}$

Compare it with $\frac{1}{u^2}$

$$\frac{1}{u+u^2} \leqslant \frac{1}{u^2} \quad \text{for all } x \in [1,\infty)$$

Since $\int_1^\infty \frac{du}{u^2}$ is convergent

$\int_1^\infty \frac{du}{u+u^2}$ is convergent.

(d) $\int_0^\infty \frac{dy}{1+e^y}$

$$\frac{1}{e^y+1} \leqslant \frac{1}{e^y} \quad \text{for all } y \in [0,\infty)$$

Since $\int_0^\infty \frac{dy}{e^y}$ is convergent therefore $\int_0^\infty \frac{dy}{1+e^y}$ is convergent.

(e) $$\int_0^\infty \frac{dx}{e^x\sqrt{x}} = \int_0^1 \frac{dx}{e^x\sqrt{x}} + \int_1^\infty \frac{dx}{e^x\sqrt{x}} \leqslant \int_0^1 \frac{dx}{\sqrt{x}} + \int_1^\infty \frac{dx}{e^x} = 2 + \frac{1}{e}$$

$\left(\frac{1}{e^x\sqrt{x}} \leqslant \frac{1}{\sqrt{x}} \text{ when } x \in (0,1] \right.$

$\left. \frac{1}{e^x\sqrt{x}} \leqslant \frac{1}{e^x} \text{ when } x \in [1,\infty) \right)$

So the original integral $\int_0^\infty \frac{dx}{e^x\sqrt{x}}$ converges.

(f) $$\int_1^\infty \frac{4\,dx}{x(x+1)}\,dx \leqslant \int_1^\infty \frac{4}{x^2}\,dx = 4$$

$\left(\frac{4}{x(x+1)} \leqslant \frac{4}{x^2} \text{ when } x \in [1,\infty) \right)$

So the integral converges.

(g) $\int_1^\infty \frac{dx}{\sqrt{x+x^3}}$

$$\frac{1}{\sqrt{x+x^3}} \leqslant \frac{1}{\sqrt{x^3}} \quad \text{for } x \in [1,\infty)$$

$\int_1^\infty \frac{dx}{x^{3/2}}$ converges therefore $\int_1^\infty \frac{dx}{\sqrt{x+x^3}}$ converges.

(h) (i) (j)

$$\int_2^\infty \frac{dx}{x(\ln x)^p}$$ let $\ln x = u$

$\frac{1}{x}dx = du$

$$\int \frac{du}{u^p} = \int u^{-p}du = \frac{1}{-p+1}u^{-p+1}$$

$$\lim_{t\to\infty} \frac{1}{-p+1}(\ln x)^{-p+1}\Big|_2^t = \lim_{t\to\infty} \frac{1}{-p+1}(\ln t)^{-p+1} - \frac{1}{-p+1}(\ln 2)^{-p+1}$$

Case 1 when $p>1$ $(\ln t)^{-p+1} \to 0$ as $t\to\infty$

Hence integral converges

$$\lim_{t\to\infty} \frac{1}{-p+1}\overset{0}{(\ln t)^{-p+1}} - \frac{1}{-p+1}(\ln 2)^{-p+1} = -\frac{1}{-p+1}(\ln 2)^{-p+1}$$

Case 2 when $p<1$ then the integral diverges.

Case 3 when $p=1$

$$\int_2^\infty \frac{dx}{x\ln x} = \int \frac{du}{u} = \lim_{t\to\infty} \ln(\ln x)\Big|_2^t$$

$$\lim_{t\to\infty} \ln(\ln t) - \ln(\ln 2) = \infty$$

Diverges.

(k, l, m) Similar argument as the above problem.

5

9 a $\lim_{x \to 0^+} x \ln x$ $(0 \cdot (-\infty))$

$= \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}}$ $\left(\frac{\infty}{\infty}\right)$

L'hopital

$= \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0^+} -x = \boxed{0}$

b $\lim_{x \to \infty} \frac{1+2x}{\sqrt{x}}$ $\left(\frac{\infty}{\infty}\right)$

$= \lim_{x \to \infty} \frac{2}{\frac{1}{2\sqrt{x}}} = \lim_{x \to \infty} 4\sqrt{x} = \infty$

c $\lim_{x \to 0^+} \frac{3^{\sin x} - 1}{x}$ $\frac{0}{0}$

$\lim_{x \to 0^+} \frac{3^{\sin x} \cdot \ln 3 \cdot \cos x}{1} = \boxed{\ln 3}$

d $\lim_{x \to \infty} e^{-x} x^{1/2}$ $0 \cdot \infty$

$= \lim_{x \to \infty} \frac{x^{1/2}}{e^x}$ $\frac{\infty}{\infty}$

$= \lim_{x \to \infty} \frac{\frac{1}{2} x^{-1/2}}{e^x} = \lim \frac{1}{2x^{1/2} e^x} = \frac{1}{\infty} = 0$

e $\lim_{x \to \infty} (\ln x)^{1/x} = a$ ∞^0

Take ln

$\lim_{x \to \infty} \ln (\ln x)^{1/x} = \ln a$

$\lim_{x \to \infty} \frac{1}{x} \ln(\ln x) = \ln a$

$0 \cdot \infty \leftarrow \lim_{x \to \infty} \frac{1}{x} \ln(\ln x)$

$\frac{\infty}{\infty}$ $\lim_{x \to \infty} \frac{\ln(\ln x)}{x} = \ln a \Rightarrow$ L'hopital $\lim_{x \to \infty} \frac{\frac{1}{\ln x} \cdot \frac{1}{x}}{1} = \ln a$

$0 = \ln a$

$\boxed{a = 1}$

(f) $\lim_{x \to 1^+} x^{\frac{1}{1-x}} = a \quad 1^\infty$

Take ln

$\Rightarrow \lim_{x \to 1^+} \ln x^{\frac{1}{1-x}} = \ln a$

$\Rightarrow \lim_{x \to 1^+} \frac{1}{1-x} \ln x = \ln a$

$\frac{0}{0} \leftarrow \lim_{x \to 1^+} \frac{\ln x}{1-x} = \ln a$

L'hopital $\lim_{x \to 1^+} \frac{\frac{1}{x}}{-1} = \ln a$

$\lim_{x \to 1^+} -\frac{1}{x} = \ln a$

$-1 = \ln a$

$\boxed{a = \frac{1}{e}}$

(10)(a) $\lim_{n \to \infty} \frac{3n}{2n-1} = 3$

(b) $\lim_{n \to \infty} \sqrt{\frac{7n}{n-4}} = \lim_{n \to \infty} \frac{\sqrt{7}\sqrt{n}}{\sqrt{n-4}} = \sqrt{7}$

(c) $\lim_{n \to \infty} \frac{4n^2 + 20n + 25}{n^2} = 4$

(d) $\lim_{n \to \infty} \frac{n^3}{e^n} \underset{\text{L'hopital}}{=} \lim_{n \to \infty} \frac{3n^2}{e^n} = \lim_{n \to \infty} \frac{6n}{e^n} = \lim_{n \to \infty} \frac{6}{e^n} = 0$

$\frac{\infty}{\infty}$

(e) $\lim_{n \to \infty} \frac{n}{10} + \frac{10}{n} = \lim_{n \to \infty} \frac{n^2 + 100}{10n} = \infty$

(f) $\lim_{n \to \infty} \frac{(-1)^n}{n} = 0$