Math 112 Fall 2009

Indeterminate forms and L'Hopital's rule

Many limits are easy to guess by inspection. Other limits are less susceptible to intuition. Consider these:

$$\lim_{x\to\infty}\frac{x^2}{2^x};\qquad \lim_{x\to 0}\frac{\sin 2x}{x};\qquad \lim_{x\to\infty} xe^{-x};\qquad \lim_{x\to\infty}\frac{x^2+1}{2x^2+3};\quad \lim_{x\to 0^+} x^x;\quad \lim_{x\to 1^+} x^{\frac{1}{1-x}}$$

Such limits are called indeterminate forms because, in each cas, two conflicting tendencies operate.

Determine the indeterminate type of the above examples: $\frac{\infty}{\infty}$, $\frac{0}{0}$, $\infty\cdot 0$, $\infty-\infty$, 1^∞, ∞^0, 0^0.

L'Hopital's rule: finding limits by differentiation

L'Hopital's rule says that **under appropriate conditions** an indeterminate form can be evaluated by differentiating the numerater and the denominator separately. In symbols:

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Applying L'Hospital (blindly): Find $\lim_{x\to 0}\frac{\sin(2x)}{x}$.

Making a point: (Blindly is bad in mathematics): Find $\lim_{x\to 0}\frac{x+4}{x+1}$.

Theorem: Let f and g be differantiable functions, such that

1. as $x\to a$, either
   (a) $f(x)\to 0$ and $g(x)\to 0$; or
   (b) $f(x)\to\pm\infty$ and $g(x)\to\pm\infty$
2. $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ exists.

Then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Note: $a\pm\infty$ and one sided limits are permitted.

General idea: Suppose f and g are differentiable functions, with $f(a)=g(a)=0$. Around $x=a$ let's write the linear approximations to f and g.

$$f(x)\approx f(a)+f'(a)(x-a)=f'(a)(x-a)$$

$$g(x)\approx g(a)+g'(a)(x-a)=g'(a)(x-a)$$

If $g'(a)\neq 0$ and $x\neq a$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(a)(x-a)}{g'(a)(x-a)} = \frac{f'(a)}{g'(a)}.$$

This should be very intuitive. The key fact here is not that f and g both go to zero at $x = a$, it is how fast they go to zero, which is measured by the rate of change, namely the derivatives of this functions at $x = a$.

Rule of Thumb: For indeterminates of the form $\frac{0}{0}$ and $\frac{\infty}{\infty}$ apply the L'Hopital's rule directly. For other cases you must rewrite the expressions to turn them into the $\frac{0}{0}$ and $\frac{\infty}{\infty}$ cases.

For indeterminates in the form of 1^∞, ∞^0 and 0^0 convert to base e exponential using the rule

$$a^b = e^{b \ln(a)}$$

and take the limit inside as in the following exercise.

Exercise: Find $\lim_{x \to \infty} (\ln(x))^{\frac{1}{x}}$.

$\lim_{x \to \infty} (\ln(x))^{\frac{1}{x}} = \lim_{x \to \infty} e^{\frac{1}{x} \ln(\ln(x)))} = e^{\lim_{x \to \infty} \frac{\ln(\ln(x))}{x}} = e^{\lim_{x \to \infty} \frac{\frac{1}{\ln(x)} \frac{1}{x}}{1}} = e^0 = 1.$