Computing Integrals using Riemann Sums and Sigma Notation
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The problems below are fairly complicated with several steps. I expect you to show your reasoning clearly and in an organized fashion. Here is the solution of a similar problem, which should give you an idea of how to write up your solution.

Example: Use left Riemann sums to compute $\int_1^3 x dx$.
Solution: $\Delta x = \frac{2}{n}$. Using left Riemann sums we get the following approximation:

$$\int_1^3 x dx \approx \frac{2}{n} \Sigma_{k=1}^n \left(1 + (k-1)\frac{2}{n}\right)$$

If you are not comfortable writing the above formula, go ahead and write the areas of the first few rectangles to see the pattern.
Let $f(x) = x$ and $\Delta x = \frac{2}{n}$.

$$\text{Area of the first rectangle} = f(1) * \frac{2}{n}$$
$$\text{Area of the second rectangle} = f(1 + \frac{2}{n}) * \frac{2}{n}$$
$$\text{Area of the } \textbf{third} \text{ rectangle} = f(1 + \mathbf{2} * \frac{2}{n}) * \frac{2}{n}$$
$$\text{Area of the } \textbf{fourth} \text{ rectangle} = f(1 + \mathbf{3} * \frac{2}{n}) * \frac{2}{n}$$
$$\vdots$$
$$\text{Area of the } k\textbf{th} \text{ rectangle} = f(1 + (\mathbf{k} - \mathbf{1}) * \frac{2}{n}) * \frac{2}{n}$$

Now to find the exact value of the integral you have to take the limit, that is

$$\int_1^3 x dx = \lim_{n\to\infty} \frac{2}{n} \Sigma_{k=1}^n \left(1 + (k-1)\frac{2}{n}\right)$$

Let us simplify the expression inside the sigma notation first.

$$\begin{aligned}
\int_1^3 x dx &= \lim_{n\to\infty} \frac{2}{n} \Sigma_{k=1}^n \left(1 + (k-1)\frac{2}{n}\right) && (1)\\
&= \lim_{n\to\infty} \frac{2}{n} \Sigma_{k=1}^n \frac{n + 2k - 2}{n} && (2)\\
&= \lim_{n\to\infty} \frac{2}{n^2} \Sigma_{k=1}^n n + 2k - 2 && (3)\\
&= \lim_{n\to\infty} \frac{2}{n^2} \left(\Sigma_{k=1}^n n + \Sigma_{k=1}^n 2k - \Sigma_{k=1}^n 2\right) && (4)\\
&= \lim_{n\to\infty} \frac{2}{n^2} \left(n^2 + 2\frac{n(n+1)}{2} - 2n\right) && (5)\\
&= \lim_{n\to\infty} \left(\frac{2n^2}{n^2} + \frac{2n^2 + 2n}{n^2} - \frac{4n}{n^2}\right) && (6)\\
&= 2 + 2 = 4 && (7)
\end{aligned}$$

Step 5 requires the formulas and properties of the sigma notation.

Now it is your turn to do the following problems.

1. Use right Riemann sums to compute the following integrals.
   (a) $\int_{-1}^{5} x dx$
   (b) $\int_{1}^{2} x^2 dx$

2. Use left Riemann sums to compute the following integrals.
   (a) $\int_{-1}^{3} x dx$
   (b) $\int_{2}^{4} x^2 dx$

3. Use midpoint Riemann sum to compute the following integrals.

   (a) $\int_1^2 x dx$

   (b) $\int_0^2 x^2 dx$

4. Use right Riemann sums to compute the following integral. $\int_{-1}^5 2x^2 + 3x dx$

Now we approach the problem from the other direction, namely given a limit of a Riemann sum, we will write that as a definite integral and compute the integral using FTC.

1. Evaluate $\lim_{n\to\infty} \Sigma_{i=1}^{n} \left(\frac{i}{n}\right)^3$ by expressing it as a definite integral and then evaluating this integral.

2. Evaluate $\lim_{n\to\infty} \Sigma_{i=1}^{n} \frac{1}{n} \left(\frac{n+i}{n}\right)^2$ by expressing it as a definite integral and then evaluating this integral.

3. Evaluate $\lim_{n\to\infty} \frac{4}{n} \Sigma_{i=1}^{n} \sqrt{\left(\frac{2(2i-1)}{n}\right)}$ by expressing it as a definite integral and then evaluating this integral.