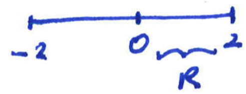
Solutions 11.5

3. $\sum_{j=1}^{\infty}\left(\frac{x}{2}\right)^{j}$ $\quad \lim_{j\to\infty}\frac{\left|\left(\frac{x}{2}\right)^{j+1}\right|}{\left|\left(\frac{x}{2}\right)^{j}\right|} = \lim_{j\to\infty}\frac{\left|\frac{x}{2}\right|^{j+1}}{\cancel{\left|\frac{x}{2}\right|^{j}}} = \lim_{j\to\infty}\left|\frac{x}{2}\right| = \frac{|x|}{2} < 1$

$|x| < 2$ so the radius



4. $\sum_{k=1}^{\infty}\frac{x^k}{k\,3^k}$ $\quad \lim_{k\to\infty}\frac{\left|\frac{x^{k+1}}{(k+1)(3^{k+1})}\right|}{\left|\frac{x^k}{k\,3^k}\right|} = \lim_{k\to\infty}\frac{|x|^{\cancel{k+1}}}{(k+1)\,3^{\cancel{k+1}}}\cdot\frac{k\,\cancel{3^k}}{\cancel{|x|^k}}$

$= \lim_{k\to\infty}\frac{k\,|x|}{3k+3} = \frac{|x|}{3} < 1$

$|x| < 3$ So $R = 3$

5. $\sum_{k=1}^{\infty}\frac{x^k}{\sqrt{k}}$ $\quad \lim_{k\to\infty}\frac{|x|^{\cancel{k+1}}}{\sqrt{k+1}}\cdot\frac{\sqrt{k}}{\cancel{|x|^k}} = |x|\lim_{k\to\infty}\sqrt{\frac{k}{k+1}}$

$= |x|\sqrt{\lim_{k\to\infty}\frac{k}{k+1}} = |x|$

$|x| < 1 \quad R = 1$

6. $\sum_{n=0}^{\infty}\frac{x^n}{n!+n}$ $\quad \lim_{n\to\infty}\frac{|x|^{\cancel{n+1}}}{(n+1)!+(n+1)}\cdot\frac{n!+n}{\cancel{|x|^n}} = 0 < 1$ for all values of x so $R = \infty$

For problems 3-6 we are not asked to find the interval of convergence. If so we have to check the endpoints to make sure whether the given power series converge at the endpoints of the interval.

7. $\sum_{n=0}^{\infty}(x-2)^n$ $\quad \lim_{n\to\infty}\frac{|x-2|^{n+1}}{|x-2|^n} = |x-2|$

Converges when $|x-2| < 1$, $\quad -1 < x-2 < 1$, $\quad 1 < x < 3$ $\quad$ So $R = 1$.

when $x = 1$ $\quad \sum_{n=0}^{\infty}(1-2)^n = \sum_{n=0}^{\infty}(-1)^n$ diverges

when $x = 3$ $\quad \sum_{n=0}^{\infty}(3-2)^n = \sum_{n=0}^{\infty}(1)^n$ diverges. So the interval of convergence is $(1,3)$.

8. $\sum_{n=2}^{\infty} \frac{(x-3)^{2n}}{n^4}$ $\quad \lim_{n\to\infty} \frac{|x-3|^{\cancel{2n}+2}}{(n+1)^4} \cdot \frac{(n)^4}{\cancel{|x-3|^{2n}}}$

$$= |x-3|^2 \underbrace{\lim_{n\to\infty} \frac{n^4}{(n+1)^4}}_{1} = |x-3|^2$$

converges when $|x-3|^2 < 1$

$|x-3| < 1$

$-1 < x-3 < 1$

$2 < x < 4.$ $\quad$ So $R = 1$

At the end points

$x=2$ $\quad \sum_{n=2}^{\infty} \frac{(2-3)^{2n}}{n^4} = \sum_{n=2}^{\infty} \frac{(-1)^{2n}}{n^4}$ $\quad$ (p- series with $p=4$) so converges

$x=4$ $\quad \sum_{n=2}^{\infty} \frac{(4-3)^{2n}}{n^4} = \sum_{n=2}^{\infty} \frac{1^{2n}}{n^4}$ $\quad$ (")

So the interval of convergence is $[2,4]$.

9. $\sum_{n=2}^{\infty} \frac{(x+5)^n}{n \ln n}$ $\quad \lim_{n\to\infty} \frac{|x+5|^{\cancel{n+1}}}{(n+1)\ln(n+1)} \cdot \frac{n \ln n}{\cancel{|x+5|^n}}$

$$= \lim_{n\to\infty} \frac{|x+5|\, n \ln n}{(n+1)\ln(n+1)} = |x+5| \underbrace{\lim_{n\to\infty} \frac{n \ln n}{(n+1)\ln(n+1)}}_{\frac{\infty}{\infty} \text{ L'hopital}}$$

$$\left[\lim_{n\to\infty} \frac{n \ln n}{(n+1)\ln(n+1)} = \lim_{n\to\infty} \frac{\ln n + n\, {}^1/_n}{\ln(n+1) + {}^{n+1}/_{n+1}} = \lim_{n\to\infty} \frac{\ln(n)+1}{\ln(n+1)} = \lim_{n\to\infty} \frac{{}^1/_n}{{}^1/_{n+1}} = \lim_{n\to\infty} \frac{n+1}{n} = 1 \right]$$

So $|x+5| < 1$

$-1 < x+5 < 1$

$-6 < x < -4$

At the end points

$x=-6$ $\quad \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ $\quad$ Converges by the alternating series test $\quad \left\{\frac{1}{n \ln n}\right\}$ decreasing $\quad \lim_{n\to\infty} \frac{1}{n \ln n} = 0.$

$x=-4$ $\quad \sum_{n=2}^{\infty} \frac{1}{n \ln n}$ $\quad$ diverges by the integral test

So the interval of convergence is $[-6,-4)$. $\quad R = 1$

10 $\sum_{n=1}^{\infty} \frac{(x+1)^n}{n}$ $\lim_{n\to\infty} \frac{|x+1|^{\cancel{n+1}}}{n+1} \cdot \frac{n}{\cancel{|x+1|^n}} = |x+1|$

$|x+1|<1 \Rightarrow -1<x+1<1$

$-2<x<0$ $R=1$.

When $x=-2$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ alternating harmonic series converges

$x=0$ $\sum_{n=1}^{\infty} \frac{1}{n}$ harmonic series diverges.

So the interval of convergence is $[-2, 0]$ and $R=1$.

13 $\sum_{k=1}^{\infty} \frac{x^k}{k^2 R^k}$ $\lim_{k\to\infty} \frac{|x|^{\cancel{k+1}}}{(k+1)^2 R^{\cancel{k+1}}} \cdot \frac{\cancel{R^k} k^2}{\cancel{|x|^k}} = \lim_{k\to\infty} \frac{|x| k^2}{R(k+1)^2} = \frac{|x|}{R}$

converges when $\frac{|x|}{R}<1$ so $|x|<R$

$-R<x<R$

What about the end points?

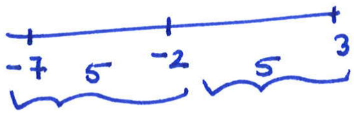
$x=-R$ $\sum_{k=1}^{\infty} \frac{(-R)^k}{k^2 R^k} = \sum_{k=1}^{\infty} \frac{(-1)^k \cancel{R^k}}{k^2 \cancel{R^k}} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ alternating p-series $p=2$ converges absolutely.

$x=R$ $\sum_{k=1}^{\infty} \frac{\cancel{R^k}}{k^2 \cancel{R^k}} = \sum_{k=1}^{\infty} \frac{1}{k^2}$ p-series, $p=2$ converges.

So the interval of convergence is $[-R, R]$.

33) $\sum_{k=0}^{\infty} a_k (x+2)^k$ converges if $x = -7$
diverges if $x = 7$

The power series is centered at $x = -2$



So it for sure converges for all values $[-7, -2]$.
Recall that the base point of a power series is the center of its interval of convergence so the power series must converge for all values in $[-7, 3)$
We don't have enough information. We also know that it diverges when $x = 7$. but it could certainly converge for $[-8, 4]$.
So the answer to 33 is it may.

34) It must

35) it may since we don't know how it behaves at that endpoint.

36) $[-11, 7)$ or $(-11, 7)$ might be possible intervals of convergence.
So it may.

37) It may.

38) It cannot.

Section 1.6

(1) $$\lim_{k\to\infty} \frac{\left|\frac{x}{2}\right|^{k+1}}{\left|\frac{x}{2}\right|^{k}} = \left|\frac{x}{2}\right| < 1$$

$$\frac{|x|}{2} < 1$$

$$-2 < x < 2 \qquad R = 2.$$

(2) $$\lim_{k\to\infty} \frac{(k+1)|x|^{\cancel{k}}}{2^{\cancel{k+1}}} \cdot \frac{\cancel{2^k}}{k\,\cancel{|x|^{k-1}}} = \frac{|x|}{2} < 1$$

$|x| < 2$ so $R=2$ as expected.

(3) $$\lim_{k\to\infty} \frac{|x|^{k+2}}{(k+2)2^{\cancel{k+1}}} \cdot \frac{\cancel{2^k}}{(k+1)|x|^{k+1}} = \frac{|x|}{2} < 1$$

$|x| < 2$ So $R=2$ as expected

(4) Recall $$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

$$f(x) = \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}$$

So $$\lim_{k\to\infty} \frac{|x|^{k+1}}{k+1} \cdot \frac{k}{|x|^k} = |x| < 1 \qquad R = 1$$

Interval of convergence $[-1, 1)$ since at -1 its the alternating hamonic series which converges.

$$f'(x) = 1 - x + x^2 - x^3 + \cdots$$
$$= \sum_{k=0}^{\infty} (-1)^k x^k$$

$$\lim_{k\to\infty} \frac{|x|^{k+1}}{|x|^k} = |x| < 1 \qquad R = 1.$$

At the endpoints $\sum (-1)^k$ and $\sum 1^k$ both diverges so its interval of convergence is $(-1, 1)$.

So both series of $f(x)$ and $f'(x)$ have the same radius of convergence but their interval of convergence differ.