Abstract II Spring 2010
Week 2
Integral Domains-Section 19

Outline:
Zero divisors of a ring
Integral domain
Characteristic of a ring

Motivation: In the ring of integers for any $a, b \in \mathbb{Z}$, $ab = 0$ implies that $a = 0$ or $b = 0$. However unlike the integers, general rings may have nonzero elements a and b whose product is zero.
Examples: $\mathbb{Z}_n$, $\mathbb{Z}_n[x]$(ring of polynomials with coeefficents in $\mathbb{Z}_n$)

Definition 0.1. *Let R be a ring. A nonzero element $A \in R$ is called a zero divisor if there is a nonzero element $b \in R$ such that either $ab = 0$ or $ba = 0$.*

Exercise: Find the zero divisors of $\mathbb{Z}_{12}$.

Generalize the above example. What can you say about the zero divisors of $\mathbb{Z}_n$.

Theorem 0.2. *In the ring $\mathbb{Z}_n$, the zero divisors are ....*

Proof:

Conclusion: In the ring $\mathbb{Z}_n$, the nonzero elements fall into one of these two categories. They are either .... or ....

Theorem 0.3. *The cancellation laws hold in a ring iff R has no zero divisors.*

Proof:

Definition 0.4. *A commutative ring with unity $1 \neq 0$ is called an integral domain if it has no zero divisors.*

Exercises:

1. Give examples of integral domains.

2. Is $\mathbb{Z}[\sqrt{3}]$ an integral domain?

3. Is $M_2(\mathbb{Z})$ an integral domain?

Theorem 0.5. *Every field is an integral domain.*

Proof:

Theorem 0.6. *Every finite integral domain is a field.*

Proof:

Definition 0.7. *If for a ring R a positive integer n exists such that $n \cdot a = 0$ for all $a \in R$, then the least such positive integer is called the characteristic of the ring R. If no such positive integer exists, then R is of characteristic zero.*

Examples:

Theorem 0.8. *The characteristic of an integral domain is either 0 or prime.*

Proof:

Exercises:

1. Let R be a ring with 1. Let $a \in R$ such that a has an inverse. Show that a cannot be a zero divisor.

2. Let R be a ring with 1 and suppose R has no zero divisors. Show that the only idempotents in R are 0 and 1.

Table 1: Summary of rings and their properties:

Ring	Form of Element	Unity	Commutative	Integral domain	Field	Characteristic
$\mathbb{Z}$	k	1	Yes	Yes	No	0
$\mathbb{Z}_n$, n composite						
$\mathbb{Z}_p$, p prime						
$\mathbb{Z}[x]$						
$n\mathbb{Z}, n \geq 1$						
$M_2(\mathbb{Z})$						
$M_2(2\mathbb{Z})$						
$\mathbb{Z}_i$						
$\mathbb{Z}_3[i]$						
$\mathbb{Z}[\sqrt{2}]$						
$\mathbb{Q}[\sqrt{2}]$						
$\mathbb{Z} \times \mathbb{Z}$						