

Abstract II Spring 2010
Rings of Polynomials-Section 22
Factorization of Polynomials over a Field-Section 23

$R[x]$ (the set of all polynomials with coefficients in the ring R) is itself a ring.
 R is a subring of $R[x]$.

x is called an indeterminate rather than variable, since x unlike in our usual understanding of polynomials does not take any values and is rather just an element of the ring $R[x]$.

Definition 0.1. Let R be a ring. A polynomial $f(x)$ with coefficients in R is a formal sum

$$\sum_{i=0}^n a_i x^i,$$

where $a_i \in R$.

Degree of a polynomial

Leading term of a polynomial

Leading coefficient of a polynomial

Theorem 0.2. The set $R[x]$ of all polynomials in an indeterminate x with coefficients in a ring R is a ring under polynomial addition and multiplication. If R is commutative, then so is $R[x]$, and if R has unity $1 \neq 0$ then 1 is also unity for $R[x]$.

Generalization to n indeterminates: The ring $R[x_1, x_2, \dots, x_n]$ of polynomials in indeterminates x_i can be defined inductively.

Proposition 0.3. Let R be an integral domain. Then

1. $\text{degree } p(x)q(x) = \text{degree } p(x) + \text{degree } q(x)$ if $p(x), q(x)$ are nonzero.
2. the units of $R[x]$ are the units of R .
3. $R[x]$ is an integral domain.

Proof:

Question True or false: If F is a field then $F[x]$ is a field.

As in Section 21, one can construct the field of quotients, $F(x)$ of $F[x]$. Any element of $F(x)$ can be represented as a quotient $f(x)/g(x)$ of two polynomials in $F[x]$ with $g(x) \neq 0$.

The Evaluation Homomorphisms

Theorem 0.4. *Let F be a subfield of a field E and let α be any element of E , and let x be an indeterminate. The map*

$$\Phi_\alpha: F[x] \rightarrow E$$

defined by

$$\Phi_\alpha(a_0 + a_1x + \cdots + a_nx^n) = a_0 + a_1\alpha + \cdots + a_n\alpha^n$$

is a homomorphism. Also $\Phi_\alpha(x) = \alpha$, and Φ_α maps F isomorphically by the identity map, i.e. $\Phi_\alpha(a) = a$ for $a \in F$. This homomorphism is called evaluation at α .

This theorem will help us to use our known language from math classes such as:

Given $p(x) = x^2 + x - 6 \in \mathbb{Q}[x]$ find $p(2)$.

In our sophisticated abstract algebra language this is basically asking to evaluate the image of $p(x)$ under the evaluation homomorphism $\Phi_2(p(x)): \mathbb{Q}[x] \rightarrow \mathbb{R}$ as defined in the above theorem.

With this new language solving a polynomial equation means finding a zero of a polynomial.

Theorem 0.5. *With the notation as above let $\Phi_\alpha: F[x] \rightarrow E$ be the evaluation homomorphism. Let*

$$f(\alpha) = \Phi_\alpha(f(x)) = a_0 + a_1\alpha + \cdots + a_n\alpha^n$$

If $f(\alpha) = 0$, then α is a zero of $f(x)$.

Factorization of Polynomials over a Field-Section 23

Throughout this section F denotes a field.

Theorem 0.6. (*Division algorithm for $f[x]$*) Let

$$f(x) = a_n x^n + \cdots + a_1 x + a_0$$

and

$$g(x) = b_m x^m + \cdots + b_1 x + b_0$$

be two elements of $F[x]$, with a_n and $b_m \neq 0$ and $m > 0$. Then there are unique polynomials $q(x)$ and $r(x)$ in $F[x]$ such that

$$f(x) = g(x)q(x) + r(x)$$

where either $r(x) = 0$ or the degree of $r(x)$ is less than the degree m of $g(x)$.

Proof:

Example:

Corollary 0.7. (*Factor theorem*) An element $a \in F$ is a zero of $f(x) \in F[x]$ iff $x - a$ is a factor of $f(x) \in F[x]$.

Proof:

Corollary 0.8. A nonzero polynomial $f(x) \in F[x]$ of degree n has at most n zeros in a field F .

Proof: