**Abstract II Spring 2010**

**The field of quotients of an integral domain-Section 21**

**Goal: To show that every integral domain can be regarded as being contained in a certain field, a field of quotients of the integral domain**

In general a commutative ring $R$ is always a subring of a larger ring $Q$ in which every nonzero element of $R$ that is not a zero divisor is a unit in $Q$. The main application of this is when $R$ is an integral domain, in which case the ring $Q$ will be a field called its field of fractions or field of quotients or quotient field.

The construction of $Q$ from $R$ will be very familiar since it follows closely the construction of the field or rational numbers $\mathbb{Q}$ from the integral domain $\mathbb{Z}$.

**Construction:**
Let $D$ be an integral domain that we desire to enlarge to a field of quotients $F$.

1. Define what the elements of $F$ are to be.
2. Define the binary operations $+$ and $\cdot$ on $F$.
3. Check all the field axioms to show that $F$ is a field under these operations.
4. Show that $F$ can be viewed as containing $D$ as an integral subdomain.

**Step 1.** Define $F$ to be the set of all equivalence classes $([a, b])$ for $a, b \in D, b \neq 0$.

**Step 2.** Define

$$[(a, b)] + [(c, d)] = [(ad + bc, bd)]$$

and

$$[(a, b)][(c, d)] = [(ac, bd)].$$

Show that these give well-defined operations of addition and multiplication on $F$.

**Step 3.** Show that $F$ is indeed a field.

**Step 4.** Show that $F$ can be regarded as containing $D$. To do that:

Show that there is an isomorphism of $D$ with a subdomain of $F$.

**Lemma 0.1.** *The map* $i\colon D \to F$ *given by* $i(a) = [(a, 1)]$ *is an isomorphism of* $D$ *with a subring of* $F$.

**Proof:**

**Uniqueness: Any two field of quotients of $D$ are isomorphic.**

**Theorem 0.2.** *Let* $F$ *be a field of quotients of* $D$ *and let* $L$ *be any field containing* $D$. *Then there exists a map* $\rho\colon F \to L$ *that gives an isomorphism of* $F$ *with a subfield of* $L$ *such that* $\rho(a) = a$ *for* $a \in D$.

**Proof:**