Abstract II Spring 2010
Homomorphisms and Factor Rings-Section 26

Recall the definition of a ring homomorphism.

Properties of Homomorphisms

Theorem 0.1. *Let R and R' be rings. Let $\Phi\colon R \to R'$ be a ring homomorphism.*

1. *If 0 is the additive identity in R then $\Phi(0)$ is the additive identity in R'.*
2. *If $a \in R$ then $\Phi(-a) = -\Phi(a)$.*
3. *If S is a subring of R, then $\Phi(S)$ is a subring of R'.*
4. *If R has unity 1, the $\Phi(1)$ is unity for $\Phi(R)$.*

Proof.

Recall the definition of the kernel of a ring homomorphism.

Theorem 0.2. *A ring homomorphism $\Phi\colon R \to R'$ is a one-to-one map iff* $\ker(\Phi) = 0$.

Proof.

Theorem 0.3. *Let $\Phi\colon R \to R'$ be a ring homomorphism with kernel H. Then the additive cosets of H form a ring R/H whose binary operations are defined by choosing representatives, i.e.*

$$(a + H) + (b + H) = (a + b) + H,$$

and

$$(a + H)(b + H) = ab + H.$$

Also the map $\mu\colon R/H \to \Phi[R]$ defined by $\mu(a + H) = \Phi(a)$ is a ring isomorphism.

Proof.

Definition 0.4. *An additive subgroup N of a ring R satisfying the properties*

$$aN \subseteq N \text{ and } Nb \subseteq N \text{ for all } a, b \in R$$

is an ideal.

Examples:

Corollary 0.5. *Let R be a ring and let I be an ideal of R. Then the additive quotient group R/I is a ring under the binary operations*

$$(r + I) + (s + I) = (r + s) + I,$$

and

$$(r + I)(s + I) = rs + I.$$

Definition 0.6. *When I is an ideal of R the ring R/I with the operations in the previous corollary is called the quotient ring of R by I.*

Theorem 0.7.

1. *(The first isomorphism theorem for rings) If $\rho\colon R \to S$ is a ring homomorphism, then $\ker(\rho)$ is an ideal of R, $\rho(R)$ is a subring of S and $R/\ker(\rho)$ is isomorphic to $\rho(R)$ as rings.*
2. *If I is any ideal of R, then the map*

$$R \to R/I \text{ defined by } r \mapsto r + I$$

is a surjective ring homomorphism with kernel I. (natural projection of R)

Proof.