Abstract II Spring 2010
Euclidean Domains-Section 46

Definition 0.1. *A euclidean norm on an integral domain D is a function ν mapping the nonzero elements of D into the nonnegative integers such that the following conditions are satisfied:*

1. *For all $a, b \in D$ with $b \neq 0$, there exists q and r in D such that $a = bq + r$ where either $r = 0$ or $\nu(r) \leq \nu(b)$.*
2. *For all $a, b \in D$, where neither a nor b is 0, $\nu(a) \leq \nu(ab)$.*

An integral domain D is a Euclidean domain if there exists a Euclidean norm on D.

Examples: 1) $\mathbb{Z}$ is a ED with $\nu(n) = |n|$. 2) If F is a field then $F[x]$ is a ED with $\nu(f(x)) = \deg f(x)$.

Theorem 0.2. *Every ED is a PID.*

Proof:

field → ED → PID → UFD → integral domain

The concept of a Euclidean norm help us to characterize the units in an integral domain.

Theorem 0.3. *For a ED with Euclidean norm ν, $\nu(1)$ is minimal among all $\nu(a)$ for nonzero $a \in D$, and $u \in D$ is a unit if and only if $\nu(u) = \nu(1)$.*

Proof:

Examples:

Theorem 0.4. *(Euclidean Algorithm) Let D be a Euclidean domain with a Euclidean norm ν, and let a and b be nonzero elements of D. Let r_1 be as in Condition 1 for a Euclidean norm, i.e*

$$a = bq_1 + r_1,$$

where either $r_1 = 0$ or $\nu(r_1) < \nu(b)$. If $r_1 \neq 0$, let r_2 be such that

$$b = r_1 q_2 + r_2,$$

where either $r_2 = 0$ *or* $\nu(r_2) < \nu(r_1)$. *In general let* r_{i+1} *be such that*

$$r_{i-1} = r_i q_{i+1} + r_{i+1},$$

where either $r_{i+1} = 0$ *or* $\nu(r_{i+1}) < \nu(r_i)$. *Then the sequence* $r_1, r_2, \ldots$ *must terminate with some* $r_s = 0$. *If* $r_1 = 0$ *then* b *is a gcd of* a *and* b. *If* $r_1 \neq 0$ *and* r_s *is the first* $r_i = 0$, *then a gcd of* a *and* b *is* r_{s-1}.
Furthermore if d *is a gcd of* a *and* b, *then there exist* λ *and* μ *in* D *such that* $d = \lambda a + \mu b$.

Let us first look at an example to digest this theorem before we proceed with the proof.
Example:

Proof: