

**Abstract II Spring 2010**  
**Extension Fields-Section 29 Part 1**

**Goal:** Every nonconstant polynomial can find a home to live where it has a zero :)

**Definition 0.1.** *A field  $E$  is an extension field of a field  $F$  if  $F$  is contained in  $E$  and is denoted by  $E/F$ . The field  $F$  is sometimes called the base field of the extension.*

**Theorem 0.2 (Kronecker's Theorem).** *Let  $F$  be a field and let  $f(x)$  be a nonconstant polynomial in  $F[x]$ . Then there exists an extension field  $E$  of  $F$  and  $\alpha \in E$  such that  $f(\alpha) = 0$ .*

**Proof:**

**Examples:**

**Definition 0.3.** *An element  $\alpha$  of an extension field  $E$  of a field  $F$  is algebraic over  $F$  if  $f(\alpha) = 0$  for some nonzero  $f(x) \in F[x]$ . If  $\alpha$  is not algebraic over  $F$ , then  $\alpha$  is transcendental over  $F$ .*

**Examples:**

**Definition 0.4.** An element of  $\mathbb{C}$  that is algebraic over  $\mathbb{Q}$  is an algebraic number. A transcendental number is an element of  $\mathbb{C}$  that is transcendental over  $\mathbb{Q}$ .

**Theorem 0.5.** Let  $E$  be an extension field of a field  $F$  and let  $\alpha \in E$ . Let  $\phi_\alpha: F[x] \rightarrow E$  be the evaluation homomorphism of  $F[x]$  into  $E$  such that

$$\phi_\alpha(a) = a \text{ for } a \in F \text{ and } \phi_\alpha(x) = \alpha.$$

Then  $\alpha$  is transcendental over  $F$  if and only if  $\phi_\alpha$  gives an isomorphism of  $F[x]$  with a subdomain of  $E$ , that is if and only if  $\phi_\alpha$  is a one-to-one map.

**Proof:**