Abstract II Spring 2010
Wednesday 01/20
Rings and Fields

Review of the last lecture:
Definition of a ring
Examples of rings
Know examples of commutative rings with identity/noncommutative rings with identity/commutative rings without identity/noncommutative rings without identity.
The following theorem allows us to use the usual rules for signs.

Theorem 0.1. *Let R be a ring. Then*

1. $0a = 0$
2. $a(-b) = (-a)b = -(ab)$
3. $(-a)(-b) = ab$

Proof. In proofs like this where additive and multiplicative concepts of a ring appear together think about ways of using the distributive laws. □

Outline of today's lecture

1. Ring homomorphisms
2. Multiplicative inverses, units of a ring
3. Division rings and fields
4. Subring and subfield

Definition 0.2. *1. Let R and R' be two rings. A map $\Phi\colon R \to R'$ is a (ring) homomorphism if the following conditions are satisfied for all $a, b \in R$:*

(a) $\Phi(a+b) = \Phi(a) + \Phi(b)$
(b) $\Phi(ab) = \Phi(a)\Phi(b)$.

2. The kernel of Φ, $\ker \Phi = \{a \in R | \Phi(a) = 0'\}$.

3. A bijective ring homomorphism is called an isomorphism.

Exercises Decide whether the mappings below are ring homomorphisms. If so prove it, if not say why not.

1. Let F be the ring of all functions from $\mathbb{R}$ to $\mathbb{R}$. For each $a \in \mathbb{R}$ define

$$\Phi_a\colon F \to \mathbb{R},$$

where $\Phi_a(f) = f(a)$ for $f \in F$.
Note this is called **evaluation homomorphism**.

2. For $n \in \mathbb{Z}$, $\rho_n : \mathbb{Z} \to \mathbb{Z}$ defined by $\rho_n(x) = nx$.

3. The function $f : \mathbb{Z}_6 \to \mathbb{Z}_{10}$ defined by $f([a]) = 5([a])$ for all $[a] \in \mathbb{Z}_6$.

Intro to Fields
Recall that not every ring is commutative or have an identity(multiplicative).
Some remarks:
The multiplicative identity of a ring is denoted by 1 and is called **unity**.
The only ring where 0 acts as both the multiplicative and the additive identity is the zero ring. So from now on whenever we talk about a ring with unity we assume $1 \neq 0$.
The multiplicative inverse for an element $a \in R$ with 1 is an element $a^{-1} \in R$ such that $aa^{-1} = a^{-1}a = 1$.
Elements that have multiplicative inverses are called units.

Example: Find the units of $\mathbb{Z}_6$ and $\mathbb{Z}_7$.

Exercise: Let R be a ring with 1 and T be the set of all units of R. Then

1. $T \neq 0$
2. 0 is not in T
3. $ab \in$ for all $a, b \in T$.

Proof:

Definition 0.3. *1. A ring R with 1 is called a division ring if every nonzero element of R is a unit.*

2. A field is a commutative division ring.

Give examples of fields.

Definition 0.4. *A subring of a ring is a subset of the ring that is a ring under induced operations from the whole ring.*

Theorem 0.5. *Let R be a ring. A nonempty subset R' of R is a subring iff $x - y \in R'$ and $xy \in R'$ for all $x, y \in R'$.*

Proof. Part 1: Assume R' is a subring.

Part 2: Suppose $x - y \in R'$ and $xy \in R'$ for all $x, y \in R'$.

□

Exercise: Let $(A, +)$ be an abelian group. Let $\text{End}(A)$ denote the set of endomorphisms of the group A into itself. Define addition and multiplication on in $\text{End}(A)$ by: $(f + g)(a) = f(a) + g(a)$ and $(f \circ g)(a) = f(g(a))$, for all $f, g \in \text{End}(A)$ and $a \in A$.
Show that $(\text{End}(M), +, \circ)$ is a ring.

Exercise: Let $f\colon R \to S$ be a ring homomorphism. Then

1. The set $\operatorname{Im} f = \{f(a) | a \in R\}$ is a subring of S.

Exercise: Let $f\colon R \to S$ be a ring homomorphism. Then $\ker f = (0)$ iff f is 1-1.