Math 112 – Exam 2
Spring 2009 – Marian Frazier

Name SOLUTIONS

Directions: Please answer all of the questions contained within. Make sure your answers are clearly identified. Once you have an answer in terms of numbers only, you need not simplify further. No books, notes, calculators, or computers are allowed on any portion of this exam. The exam is worth 100 points. You must show all work to receive credit. *Good luck!*

The error bound formula from Taylor's Theorem is:

$$|P_n(x) - f(x)| \le \frac{K_{n+1}|x - x_0|^{n+1}}{(n+1)!}$$

And in case you need them to help you through any antidifferentiation:
Half-angle formulas:

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x)) \qquad \cos^2(x) = \frac{1}{2}(1 + \cos(2x))$$

Double-angle formulas:

$$\sin(2x) = 2\sin x \cos x \qquad \cos(2x) = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

Question	Score	Maximum
1		16
2		15
3		11
4		15
5		16
6		15
7		12
bonus		+5
Total		100

SOLUTIONS

14 pts.

1. Determine whether or not the improper integral converges. If it converges, evaluate the integral. If it diverges, specify whether the integral diverges to ∞, $-\infty$, or neither.

8 pts.

a. $\int_0^{\pi/2} \tan x\,dx = \int_0^{\pi/2} \frac{\sin x}{\cos x}\,dx$, which is discontinuous at $\cos x = 0 \Rightarrow x = \pi/2$ +1

$= \lim_{t\to\pi/2^-} \int_0^t \frac{\sin x}{\cos x}\,dx$ +1 Let $u = \cos x$, $du = -\sin x\,dx$ +1

$= \lim_{t\to\pi/2^-} -\ln u \Big]_{x=0}^{x=t}$ +1 $= \lim_{t\to\pi/2^-} \left(-\ln(\cos x)\right)\Big]_0^t$ +1

$= \lim\left(-\ln(\cos t) + \ln(\cos 0)\right) = -(-\infty) = \infty$ +2

- $\cos(0) = 1$, so $\ln(\cos(0)) = \ln(1) = 0$.
- As $t \to \pi/2^-$, $\cos t \to 0$ from the right. As $\cos t \to 0^+$, $\ln(\cos t) \to \ln(0) \to -\infty$.

8 pts.

b. $\int_5^{\infty} \frac{7}{x^2+3x-10}\,dx = \int_5^{\infty} \frac{7}{(x-2)(x+5)}\,dx = \int_5^{\infty} \frac{A}{x-2} + \frac{B}{x+5}\,dx = I$ +1

$7 = A(x+5) + B(x-2)$

$x = -5:\ 7 = -7B \Rightarrow B = -1$ +1

$x = 2:\ 7 = 7A \Rightarrow A = 1$ +1

So $I = \lim_{t\to\infty} \int_5^t \frac{1}{x-2} - \frac{1}{x+5}\,dx$ +1

$= \lim\left(\ln|x-2| - \ln|x+5|\right)\Big]_5^t$ +1

$= \lim_{t\to\infty}\left(\ln|t-2| - \ln|t+5| - \ln|5-2| + \ln|5+5|\right)$

+1.5

$= \lim_{t\to\infty}\left(\ln\left|\frac{t-2}{t+5}\right| + \ln\left(\frac{10}{3}\right)\right) = \ln\left(\frac{10}{3}\right)$ +1.5

- As $t\to\infty$, $\frac{t-2}{t+5} \to 1$, so $\ln\left(\frac{t-2}{t+5}\right) \to \ln(1) = 0$.

15 pts.

2. Find a proper integral whose value approximates that of the given improper integral to within 10^{-5}.

$$\int_0^\infty \frac{\arctan x}{(1+x^2)^3}\,dx \;= I$$

Note: $\arctan(x)$ lies between 0 and $\pi/2$ for all $x \geq 0$.

+2 Find value 'a' such that $\displaystyle\int_a^\infty \frac{\arctan x}{(1+x^2)^3}\,dx < 10^{-5}$.

We know that:

- (+1) $0 \leq \arctan x \leq \pi/2$ for all $x \geq 0$
- (+2) $(1+x^2)^3 > (x^2)^3 \Rightarrow \dfrac{1}{(1+x^2)^3} < \dfrac{1}{(x^2)^3} = \dfrac{1}{x^6}$ for all $x \geq 0$

+3 So $\dfrac{\arctan x}{(1+x^2)^3} < \dfrac{\pi/2}{x^6}$ for all $x \geq 0$.

$$\Rightarrow \int_a^\infty \frac{\arctan x}{(1+x^2)^3}\,dx < \int_a^\infty \frac{\pi/2}{x^6}\,dx = \frac{\pi}{2}\lim_{t\to\infty}\int_a^t \frac{1}{x^6}\,dx$$

$$= \frac{\pi}{2}\lim\left[-\frac{1}{5}x^{-5}\right]_a^t = \frac{\pi}{2}\lim_{t\to\infty}\left(-\frac{1}{5}\underset{\underset{0}{\downarrow}}{t^{-5}} + \frac{1}{5}a^{-5}\right)$$

+4 for solving indef. int.

$$= \frac{\pi}{2}\cdot\frac{1}{5}a^{-5} \overset{+1}{<} 10^{-5}$$ ~ Find 'a' that satisfies this inequality.

$$\Rightarrow a^{-5} < \frac{10\cdot 10^{-5}}{\pi} = \frac{10^{-4}}{\pi}$$

$$\Rightarrow a^5 > \frac{\pi}{10^{-4}} = 10^4\pi$$

$$\Rightarrow a > (10^4\pi)^{1/5}$$

+3

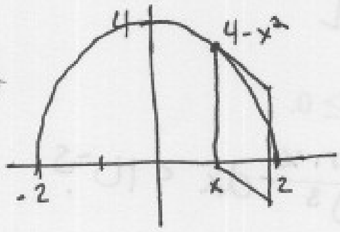
So $\displaystyle\int_0^{(10^4\pi)^{1/5}} \frac{\arctan x}{(1+x^2)^3}\,dx$ should approximate I to within 10^{-5}.

+2

11 pts.

3. The base of a solid is the region bounded by the graphs of $y = 4 - x^2$ and $y = 0$. Cross-sections of the solid perpendicular to the x-axis are squares. Find the volume of the solid.

×2



×3 $A(x) = (\text{side})^2 = (4-x^2)^2$

$$V = \int A(x)\,dx$$

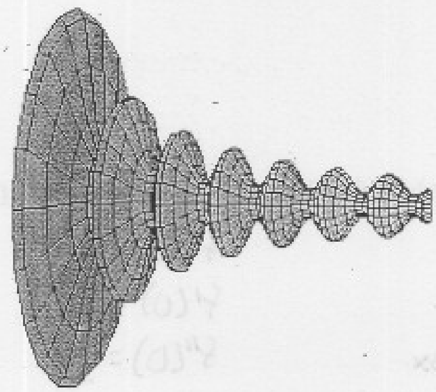
×3 $$= \int_{-2}^{2} (4-x^2)^2\,dx$$

+8 for integral

$$V = \int_{-2}^{2} (16 - 8x^2 + x^4)\,dx = \left(16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right]_{-2}^{2}$$

$$= 16(2) - \frac{8}{3}(8) + \frac{32}{5} - \left(-32 - \frac{8}{3}(-8) + \frac{-32}{5}\right)$$

$$= 2\cdot 32 - 2\cdot\frac{64}{3} + 2\cdot\frac{32}{5} = \frac{512}{15}$$



15 pts.

4. Let V be a solid created by rotating the curve

$$f(x) = \frac{\sin x + 2}{x}$$

around the x axis over the interval $[2\pi, \infty)$, as shown above. Does this solid have finite or infinite volume?

"Slices" of this solid will be circles with radius $f(x)$. Thus, the volume of the solid can be found by integrating the area, $\pi(f(x))^2$, over the interval.

x3

$$V = \int_{2\pi}^{\infty} \pi \left(\frac{\sin x + 2}{x}\right)^2 dx = \pi \lim_{t\to\infty} \int_{2\pi}^{t} \frac{(\sin x + 2)^2}{x^2}\, dx$$

x3

Now since $-1 \le \sin x \le 1$, $1 \le \sin x + 2 \le 3$, and $1 \le (\sin x + 2)^2 \le 9$.

x2

$$\text{So } V \le \pi \lim_{t\to\infty} \int_{2\pi}^{t} \frac{9}{x^2}\, dx = 9\pi \lim_{t\to\infty} \left[-\frac{1}{x}\right]_{2\pi}^{t}$$

x2

$$= 9\pi \lim_{t\to\infty} \left(-\frac{1}{t} + \frac{1}{2\pi}\right) = \frac{9\pi}{2\pi} = \frac{9}{2}$$

+3 for integral

+2

So the solid has finite volume; specifically, volume less than $9/2$.

16 pts. 8 pts.

5. a. Find the fifth-order Maclaurin polynomial for $f(x) = e^{3x}$.

$f(x) = e^{3x}$ $\quad f(0) = e^0 = 1$

$f'(x) = 3e^{3x}$ $\quad f'(0) = 3$

$f''(x) = 3^2 e^{3x}$ $\quad f''(0) = 3^2$

$f'''(x) = 3^3 e^{3x}$ $\quad f'''(0) = 3^3$

$f^{(4)}(x) = 3^4 e^{3x}$ $\quad f^{(4)}(0) = 3^4$

$f^{(5)}(x) = 3^5 e^{3x}$ $\quad f^{(5)}(0) = 3^5$

So $M_5(x) = 1 + 3x + \frac{3^2}{2!}x^2 + \frac{3^3}{3!}x^3 + \frac{3^4}{4!}x^4 + \frac{3^5}{5!}x^5$

8 pts.

b. Write in sigma notation the 978th order Taylor polynomial based at 1 for $f(x) = e^{3x}$.

Using the same derivatives as above, but now $x_0 = 1$.

$f(1) = e^3$

$f'(1) = 3e^3$

$f''(1) = 3^2 e^3$

$f'''(1) = 3^3 e^3$

$\vdots$

$f^{(n)}(1) = 3^n e^3$

So $P_{978}(x) = \sum_{i=0}^{978} \frac{3^i e^3}{i!}(x-1)^i$

15 pts.

6. Let $M_n(x)$ be the nth-degree Macluarin polynomial for $h(x) = \cos(2x)$. Find an integer n such that

$$|M_n(1/2) - \cos(1)| < 0.001.$$

Hint: You only care about the maximum possible error achieved by $M_n(x)$ when x is in the interval $I = [0, 1/2]$. $x_0 = 0$

Note: $h(x) = \cos(2x)$
$h'(x) = -2\sin(2x)$
$h''(x) = -4\cos(2x)$
$h'''(x) = 8\sin(2x)$

Since $\sin(2x)$ and $\cos(2x)$ are always between -1 and 1, we can say that

$$|h^{(n+1)}(x)| \leq |\pm 2^{n+1}| = 2^{n+1} = K_{n+1}$$

Then according to Taylor's Thm,

$$|M_n(1/2) - \cos(2 \cdot 1/2)| < \frac{2^{n+1}|1/2 - 0|^{n+1}}{(n+1)!} = \frac{2^{n+1} \cdot \left(\frac{1}{2}\right)^{n+1}}{(n+1)!} = \frac{1}{(n+1)!}$$

We must find n such that

$$\frac{1}{(n+1)!} < 0.001 = \frac{1}{1000}$$

$$(n+1)! > 1000$$

Guess: $1! = 1$
$2! = 2$
$3! = 6$
$4! = 24$
$5! = 120$
$6! = 720$
$7! = 5040$

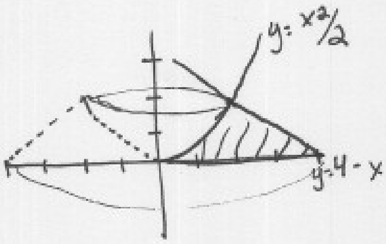
So take $n = 6$.

12 pts.

7. Let S be the region bound by the curves

$$y = \frac{x^2}{2}, y = 4 - x, y = 0.$$

Find the volume of the solid created when S is rotated around the y-axis.



$y = 4 - x \Rightarrow x = 4 - y = R(y)$

$y = x^2/2 \Rightarrow x = \sqrt{2y} = r(y)$

$V = \pi \int R^2(y) - r^2(y)\, dy$

$= \pi \int_0^2 (4-y)^2 - (\sqrt{2y})^2 dy = \pi \int_0^2 (16 - 8y + y^2 - 2y) dy$

$= \pi \left(16y - 5y^2 + \frac{1}{3}y^3\right]_0^2 = \pi\left(32 - 20 + \frac{8}{3}\right) = \frac{44\pi}{3}$

BONUS 5 pts.

Find the center of mass of the solid of rotation described in Problem 7.

$\bar{y} = \dfrac{\pi \int_0^2 y(16 - 10y + y^2)dy}{\pi \int_0^2 (16 - 10y + y^2)dy} = \dfrac{A}{B}$ we know $B = \frac{44\pi}{3}$

$A = \pi\left(8y^2 - \frac{10}{3}y^3 + \frac{1}{4}y^4\right]_0^2 = \pi\left(32 - \frac{80}{3} + \frac{16}{4}\right) = \pi\left(\frac{108}{3} - \frac{80}{3}\right) = \frac{28\pi}{3}$

So $\bar{y} = \dfrac{28\pi/3}{44\pi/3} = \dfrac{28}{44} = \dfrac{7}{11}$