Math 112 – Exam 1
Spring 2009 – Marian Frazier

Name Solutions

Directions: Please answer all of the questions contained within. Make sure your answers are clearly identified. No books, notes, calculators, or computers are allowed on any portion of this exam. The exam is worth 90 points; point values for each problem are indicated in parentheses. You must show all work to receive credit. Good luck!

Error bound formulas:

$$|I-L_n| \le \frac{K_1(b-a)^2}{2n} \qquad |I-R_n| \le \frac{K_1(b-a)^2}{2n}$$

$$|I-T_n| \le \frac{K_2(b-a)^3}{12n^2} \qquad |I-M_n| \le \frac{K_2(b-a)^3}{24n^2}$$

Half-angle formulas:

$$\sin^2(x) = \frac{1}{2}(1-\cos(2x)) \qquad \cos^2(x) = \frac{1}{2}(1+\cos(2x))$$

Double-angle formulas:

$$\sin(2x) = 2\sin x\cos x \qquad \cos(2x) = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

Trig substitutions:

If the integrand involves…	…substitute
$\sqrt{a^2-x^2}$	$x = a\sin t$
$\sqrt{a^2+x^2}$	$x = a\tan t$
$\sqrt{x^2-a^2}$	$x = a\sec t$

Question	Score	Maximum
1		10
2		15
3		10
4		10
5		20
6		10
7		8
8		10
Total		90 (+3 bonus)

1. (10 pts) Prove the reduction formula for sines: (Make sure you show all your work.)

$$\int(\sin x)^n\,dx = -\frac{1}{n}\cos x(\sin x)^{n-1} + \frac{n-1}{n}\int(\sin x)^{n-2}\,dx$$

$\int(\sin x)^n dx = \int(\sin x)^{n-1}(\sin x)dx$ Let $u=(\sin x)^{n-1}$ $du=(n-1)(\sin x)^{n-2}\cos x c$

$dv=\sin x\,dx$ $v=-\cos x$

IBP: $I = -(\sin x)^{n-1}\cos x + (n-1)\int(\sin x)^{n-2}\cos^2 x\,dx$

$\int(\sin x)^{n-2}\cos^2 x\,dx = \int(\sin x)^{n-2}(1-\sin^2 x)\,dx = \int(\sin x)^{n-2} - (\sin x)^n\,dx$

So

$I = -(\sin x)^{n-1}\cos x + (n-1)\int(\sin x)^{n-2}dx - (n-1)\underbrace{\int(\sin x)^n dx}_{I}$

$(n-1)I + I = -(\sin x)^{n-1}\cos x + (n-1)\int(\sin x)^{n-2}dx$

$I = -\frac{1}{n}(\sin x)^{n-1}\cos x + \frac{n-1}{n}\int(\sin x)^{n-2}dx$ ✓

2. (15 pts) Evaluate the following integrals.

a. $\int_0^1 e^{2x} e^{e^x}\, dx$ (Note: Simplify your answer as much as possible.)

Let $t = e^x$, $dt = e^x\, dx$

$\parallel$

$$\int_{x=0}^{x=1} e^x \cdot e^x \cdot e^{e^x}\, dx = \int_{x=0}^{x=1} t e^t\, dt$$

Let $u = t$ $\quad du = dt$

$dv = e^t\, dt$ $\quad v = e^t$

Then

$$\int_{x=0}^{x=1} t e^t\, dt = t e^t \Big]_{x=0}^{x=1} - \int_{x=0}^{x=1} e^t\, dt = t e^t - e^t \Big]_{x=0}^{x=1} = e^x e^{e^x} - e^{e^x} \Big]_0^1$$

$$= e^1 e^{e^1} - e^{e^1} - e^0 e^{e^0} + e^{e^0} = e \cdot e^e - e^e - 1 \cdot e^1 + e^1$$

$$= e \cdot e^e - e^e = e^{(1+e)} - e^e = e^e(e-1)$$

b. $\int \frac{x^2}{\sqrt{9-x^2}}\, dx$

Let $x = 3\sin t$, $dx = 3\cos t\, dt$

$\sqrt{9-x^2} = \sqrt{9 - 9\sin^2 t} = 3\sqrt{1-\sin^2 t} = 3\cos t$

So $$\int \frac{x^2}{\sqrt{9-x^2}}\, dx = \int \frac{9\sin^2 t}{3\cos t} \cdot 3\cos t\, dt = 9\int \sin^2 t\, dt$$

$$= \frac{9}{2}\int (1 - \cos(2t))\, dt = \frac{9}{2}\left[t - \frac{1}{2}\sin(2t)\right] + C$$

$$= \frac{9}{2}\left[t - \frac{1}{2} \cdot 2\sin t \cdot \cos t\right] + C$$

$$= \frac{9}{2}\left[\arcsin\left(\frac{x}{3}\right) - \frac{x}{3} \cdot \frac{\sqrt{9-x^2}}{3}\right] + C$$

$$= \frac{9}{2}\arcsin\left(\frac{x}{3}\right) - \frac{x\sqrt{9-x^2}}{2} + C$$

c. $\int \frac{2x-1}{x^2-2x-3}\,dx = \int \frac{A}{x-3} + \frac{B}{x+1}\,dx$

$2x-1 = A(x+1)+B(x-3)$

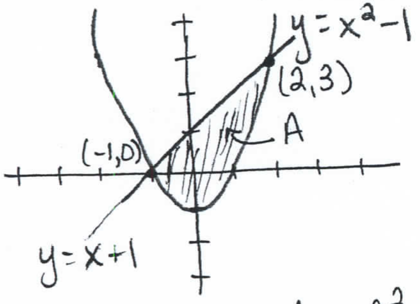
$x=-1: -3=-4B \Rightarrow B=\frac{3}{4}$

$x=3: 5=4A \Rightarrow A=\frac{5}{4}$

So $\int \frac{2x-1}{x^2-2x-3}\,dx = \frac{5}{4}\int \frac{1}{x-3}\,dx + \frac{3}{4}\int \frac{1}{x+1}\,dx$

$= \frac{5}{4}\ln|x-3| + \frac{3}{4}\ln|x+1| + C$

3. (10 pts) Find the area of the region bound by the curves $y = x^2 - 1$ and $y = x + 1$. (Note: Once you reduce everything to numbers, you need not simplify.)



pts. of intersection: $x^2-1=x+1$

$x^2-x-2=0$

$(x-2)(x+1)=0 \Rightarrow x=2$ or $x=-1$

$y=3 \qquad y=0$

$A = \int_{-1}^{2} (x+1)-(x^2-1)\,dx = \int_{-1}^{2} (x+1-x^2+1)\,dx$

$= \frac{1}{2}x^2 + 2x - \frac{x^3}{3}\Big]_{-1}^{2} = \left(\frac{4}{2}+4-\frac{8}{3}\right) - \left(\frac{1}{2}-2+\frac{1}{3}\right)$

$= \frac{3}{2} + 6 - \frac{9}{3} = \frac{9}{2}$

4. (10 pts) Set up an inequality to determine a value of n such that the trapezoid rule approximates

$$I = \int_1^3 \cos(1+x^2)\,dx$$

with error guaranteed to be less than 0.0001. Your final answer should be in the form $n \geq something$, where the something contains only numbers; but you do not need to simplify beyond that.

$f(x) = \cos(1+x^2) \qquad 1 \leq x \leq 3$

$f'(x) = -\sin(1+x^2) \cdot 2x$

$f''(x) = -\sin(1+x^2) \cdot 2 + 2x \cdot -\cos(1+x^2) \cdot 2x$

$$|f''(x)| = |-2\sin(1+x^2) + -4x^2 \cdot \cos(1+x^2)|$$

$$\leq \underbrace{|-2\sin(1+x^2)|}_{\text{always} \leq 2} + \underbrace{|-4x^2|}_{\substack{\text{always} \\ \leq 4 \cdot 9 = 36}} \cdot \underbrace{|\cos(1+x^2)|}_{\substack{\text{always} \\ \leq 1}}, \text{ by triangle inequality}$$

$$\leq 2 + 36 \cdot 1 = 38 = K_2$$

Then, using the error bound formula for T_n,

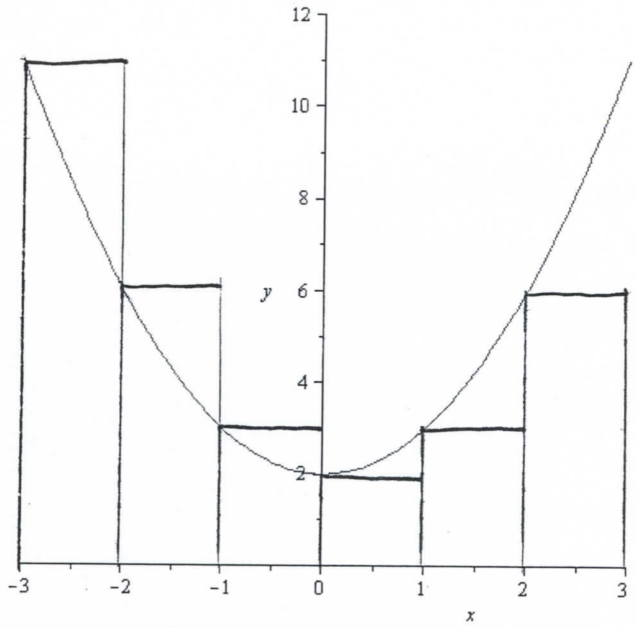
$$|I - T_n| \leq \frac{38(3-1)^3}{12n^2} = \frac{38 \cdot 8}{12n^2} \leq 0.0001 = \frac{1}{10000}$$

$$\text{So } n^2 \geq \frac{38 \cdot 8}{12(0.0001)} = \frac{38 \cdot 8 \cdot 10000}{12}$$

$$n \geq \sqrt{\frac{38 \cdot 8}{12(0.0001)}} = \sqrt{\frac{38 \cdot 8 \cdot 10000}{12}}$$

Any n larger than this values guarantees an error (using T_n) of less than 0.0001.

5. (20 pts) Consider the curve $x^2 + 2$, as seen in the graph below:



a. (7 pts) Approximate the area under this curve between x = -3 and x = 3 using a left sum approximation with 6 subintervals.

$L_6 = f(-3)\cdot 1 + f(-2)\cdot 1 + f(-1)\cdot 1 + f(0)\cdot 1 + f(1)\cdot 1 + f(2)\cdot 1$

$= 11 + 6 + 3 + 2 + 3 + 6 = 31$

b. (5 pts) Find the exact area under the curve between x = -3 and x = 3.

$\int_{-3}^{3} (x^2+2)\,dx = \frac{x^3}{3} + 2x \Big]_{-3}^{3} = \frac{27}{3} + 6 - \left(-\frac{27}{6} - 6\right)$

$= 9 + 6 + 9 + 6 = 30$

c. (3 pts) What was the error committed by the approximation in part (a)?

$|I - L_6| = |30 - 31| = 1$

d. (5 pts) For any *n*, will the midpoint sum approximation M_n underestimate or overestimate the area found in part (b)? You do not need to calculate M_n, but give a specific reason for your answer.

M_n will underestimate, since $f(x)$ is concave up between -3 and 3.

6. (10 pts) Evaluate each of the following limits.

a. $\lim_{x\to 1^+}\left(\frac{1}{x-1}\right) = \infty$

b. $\lim_{x\to 0}\left(e^x + x\right)^{1/x} = L$

$$\ln L = \lim_{x\to 0} \ln\left(e^x+x\right)^{1/x} = \lim_{x\to 0} \frac{1}{x}\ln\left(e^x+x\right)$$

$$= \lim_{x\to 0} \frac{\ln\left(e^x+x\right)}{x} = \frac{0}{0} \;!!$$

Apply L'Hôpital's rule :

$$= \lim_{x\to 0} \frac{\frac{1}{e^x+x}\cdot\left(e^x+1\right)}{1} = \lim_{x\to 0} \frac{e^x+1}{e^x+x} = \frac{e^0+1}{e^0+0} = \frac{1+1}{1+0} = 2$$

$$\Rightarrow L = e^2$$

7. (8 pts) Consider the integral

$$I = \int_{-\pi}^{\pi} \sqrt{4x^4+4}\, dx$$

Interpret I as the length of an appropriate curve.

arc length: $L = \int_a^b \sqrt{(f'(x))^2+1}\,dx = \int_{-\pi}^{\pi}\sqrt{4x^4+4}\,dx = I$

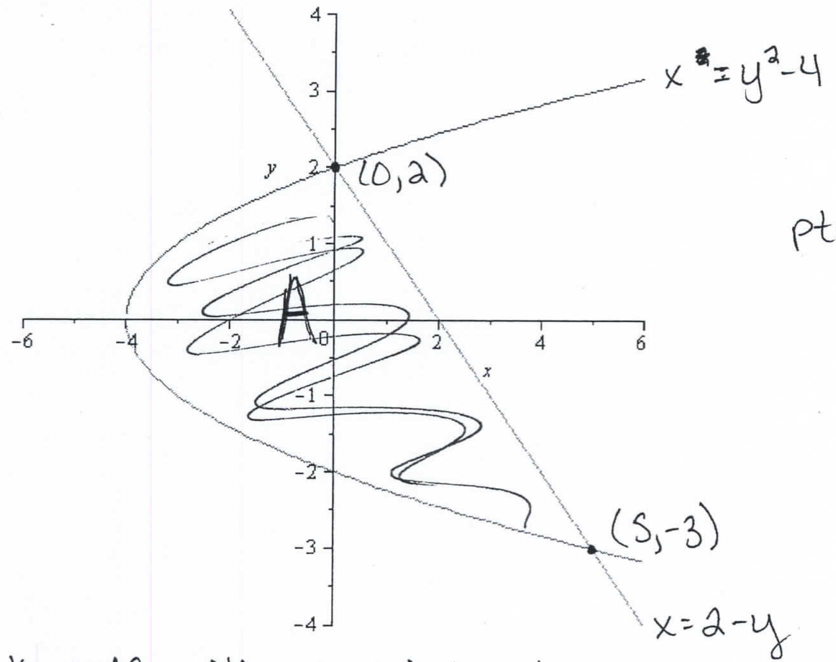
$= 2\int_{-\pi}^{\pi}\sqrt{x^4+1}\,dx$

So $(f'(x))^2 = x^4$

$f'(x) = x^2$, $f(x) = \frac{x^3}{3}$

So I is twice the length of the curve $\frac{x^3}{3}$ from $-\pi$ to π.

8. (10 pts) Find the area of the region bound by the curves $x = y^2 - 4$ and $y = 2 - x$. The graph below may assist you. (Note: Once you reduce everything to numbers, you need not simplify.)



pts. of intersection:

$y^2 - 4 = 2 - y$

$y^2 + y - 6 = 0$

$(y+3)(y-2) = 0$

$\Rightarrow y = -3$ or $y = 2$

$x = 5 \qquad x = 0$

Take area with respect to y!

$$A = \int_{-3}^{2} (2-y) - (y^2-4)\, dy = \int_{-3}^{2} (6 - y - y^2)\, dy$$

$$= 6y - \tfrac{1}{2}y^2 - \tfrac{1}{3}y^3\Big]_{-3}^{2} = \left(12 - \tfrac{4}{2} - \tfrac{8}{3}\right) - \left(-18 - \tfrac{9}{2} + \tfrac{27}{3}\right)$$

$$= 30 + \tfrac{5}{2} - \tfrac{35}{3} = \tfrac{125}{6} = 20.8333$$