Nonlinearity **10** (1997) 277–290. Printed in the UK PII: S0951-7715(97)68361-9

Viscous boundary-layer effects in nearly inviscid cylindrical flows

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Received 15 April 1996
Recommended by S Childress

Abstract. The dynamics of flow inside a cylinder at high Reynolds number are considered. A study of the viscous boundary layer near the walls is performed. In the case where there is no pressure gradient, a result is proven demonstrating that a regular perturbation expansion holds for the solution, even when a small discontinuity exists in the wall data. In addition, the characteristic decay rate of the flow in the viscous boundary layer is established. In the case where there is a pressure gradient, a result is proven demonstrating that an additional scale, related to the size of the disturbance and larger than the boundary-layer width, must be used in a multiple-scale expansion. Examination of the divergence of these multiple-scale expansions for finite disturbances leads to discussions of viscous flow and separation processes.

AMS classification scheme numbers: 76C05, 76D15, 76D30, 76F99, 76U05

1. Introduction

When modelling laminar flows at high Reynolds number, the general approach used is to model the flow away from any boundaries in an inviscid limit and then to construct viscous boundary layers to adjust the Euler flow to any boundary conditions. When modelling unbounded flows, a far-field condition is usually applied on the Euler flow, and the effects of the viscous boundary layer are localized near the boundaries.

However, in systems with closed streamlines there is no far field, so there is no way to know the properties of the Euler flow *a priori*. We consider the case of constant-vorticity Euler flow inside a cylinder [1]. Since such a system is enclosed by boundaries, the effect of the flow in the viscous boundary layer is magnified. In particular, the vorticity and Bernoulli constant of the inviscid flow are determined by interactions with the viscous boundary layer [1–4]. For instance, Wood [6] analysed several systems with closed streamlines in just such a manner. Previous work in these systems has emphasized the computation of these parameters and minimized the importance of the actual flow dynamics in the boundary layer. In this work we perform a careful analysis of the fluid flow in the boundary layer present in two-dimensional flow in a closed cylinder. Two specific cases are considered.

First, in the case where there is no pressure gradient arising from the Euler flow, a result is proven demonstrating that a regular perturbation expansion holds for the solution in the viscous region. This result is rather unexpected, given the nonlinearity of the equation, and

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holds even when there are small discontinuities in the wall data. In addition, the decay rate of the displacement field is shown to depend exponentially on the lowest Fourier mode of the wall disturbance.

In the case where there is a pressure gradient arising from the inviscid flow, the results are markedly different. First, a regular perturbation expansion is no longer appropriate; a multiple-scale expansion must be used. This expansion shows that in the viscous boundary layer, there are two typical stream scales to consider: one determined by the viscosity, and another, longer scale, determined by the disturbance.

Our work postulates an underlying relationship between the viscosity and disturbance scales: namely, that one must be transcendentally small with respect to the other. Since it does not matter which is smaller, the same mathematical framework with finite disturbances can be used to examine viscous flow and the transition to separated structures.

These results indicate that though one may be able to calculate useful quantities without solving in detail for the flow in the viscous boundary layer, such calculations miss some subtleties of the problem. In fact, in order to understand fully all the physical mechanisms inherent in such a simple flow, detailed analysis needs to be done for each of the constituent parts which play a role. The mathematical framework presented in this paper can be extended to other more complicated problems.

2. Governing equations

We wish to solve the problem of steady flow within a circular cylinder of unit radius. Here we take the case of inviscid flow, so the Reynolds number Re is very large. Away from the walls we have Euler flow. Leal [5] derives the following nondimensional equations for the boundary layer near the walls:

$$-v\frac{\partial u}{\partial y} + u\frac{\partial u}{\partial \theta} = U(\theta)U'(\theta) + \frac{\partial^2 u}{\partial y^2} \tag{2.1a}$$

$$-\frac{\partial v}{\partial y} + \frac{\partial u}{\partial \theta} = 0 \tag{2.1b}$$

with boundary conditions

$$u(0,\theta) = F(\theta) \qquad u(\infty,\theta) = U(\theta). \tag{2.2}$$

Here u is the angular velocity, v is the radial velocity scaled by $\mathrm{Re}^{1/2}$, y is the spatial variable near the wall stretched by $\mathrm{Re}^{1/2}$, $U(\theta)$ is the Euler velocity of the core evaluated at the wall, and $F(\theta)$ is the imposed wall velocity. (With a suitable change of variables, (2.1) holds for non-circular cylindrical domains.) The pressure, which in the boundary layer is a function of θ only, is given by

$$\frac{\mathrm{d}p}{\mathrm{d}\theta} = -U(\theta)U'(\theta). \tag{2.3}$$

In general, $U(\theta)$ must be determined *a posteriori*, since the boundary-layer structure will determine the constant vorticity of the Euler flow, and thereby the core velocity at the wall.

At this stage we introduce a stream function ψ defined by

$$\frac{\partial \psi}{\partial y} = \frac{u}{\sqrt{2}} \qquad \frac{\partial \psi}{\partial \theta} = \frac{v}{\sqrt{2}}$$

which immediately satisfies (2.1b). Transforming from (y,θ) coordinates to (ψ,θ) coordinates, equation (2.1a) becomes

$$\frac{1}{2}\frac{\partial (u^2 - U^2)}{\partial \theta} = \frac{1}{4}u\frac{\partial}{\partial \psi}\left(2u\frac{\partial u}{\partial \psi}\right) \tag{2.4}$$

with boundary conditions given by (2.2). Note that this transformation eliminates v from our equation.

We assume that the velocity of the viscous layer is not too different from that of the Euler flow. We simplify our equation further by introducing the *velocity displacement field* q, where $\epsilon q = u^2 - U^2$. Here $0 < \epsilon \ll 1$ is a parameter inherent in the physical system, such as the magnitude of a disturbance or the eccentricity of a nearly cylindrical body. Since it is related to the size of the displacement, we refer to ϵ as the *displacement scale*. In addition, $q = \mathrm{O}(1)$. Then letting $F^2(\theta) - U^2(\theta) = \epsilon f(\theta)$ where $f = \mathrm{O}(1)$, we have

$$2\frac{\partial q}{\partial \theta} = \sqrt{U^2 + \epsilon q}\,\frac{\partial^2 q}{\partial \psi^2} \tag{2.5}$$

with the added conditions that

$$q(\psi, 0) = q(\psi, 2\pi) \tag{2.6a}$$

$$q(0, \theta) = f(\theta) \qquad q(\infty, \theta) = 0. \tag{2.6b}$$

In a systematic treatment of any set of equations using perturbation methods, it is often expedient to treat any small or large parameters as functions of a single perturbation parameter. In this work, there are two small parameters: Re^{-1} and ϵ. Equations (2.1) and their counterpart in the core region result from taking the leading-order terms when expressing the velocity as a power series in inverse powers of Re. Taking subsequent terms in the expansion will lead to an infinite set of equations for the core and boundary-layer velocities, one at each order of Re^{-1}. Therefore, if ϵ were a power of Re^{-1} or *vice versa*, at some order the disturbance at the wall would have to be matched to the Euler flow.

We wish to avoid this complication by treating the two parameters as distinct; that is, an expansion in one will never have to be matched to the expansion in the other. Obviously, this is true only if one vanishes faster than any power of the other. Therefore, we could assume that $0 < \mathrm{Re}^{-1} \ll \epsilon \ll 1$; for instance, that $\mathrm{Re} = \mathrm{O}(\mathrm{e}^{1/\epsilon})$. By doing so, we see that for very small ϵ, in order to accept the Prandtl theory the Reynolds number would become so large that our laminar flow assumption may not be valid.

Alternatively, we could assume that $0 < \epsilon \ll \mathrm{Re}^{-1} \ll 1$; for instance, that $\epsilon = \mathrm{O}(\mathrm{e}^{-\mathrm{Re}})$. But in this case a Reynolds number large enough to inspire a boundary layer would cause ϵ to be exceedingly small. However, such disturbances may still be of physical interest. We note that the choice of assumption will not affect our mathematical analysis; therefore, our model will provide accurate mathematical results for two very different physical situations.

3. No pressure gradient

We begin by supposing that the cylinder is circular and that the fluid within it is undergoing a solid-body rotation in the interior: that is, let $U(\theta) \equiv 1$. Note from (2.3) that there is no pressure gradient. Then equation (2.5) becomes

$$2\frac{\partial q}{\partial \theta} - \sqrt{1 + \epsilon q}\,\frac{\partial^2 q}{\partial \psi^2} = 0. \tag{3.1}$$

We might expect that a regular perturbation expansion of the form

$$q(\psi, \theta; \epsilon) = \sum_{j=0}^{\infty} \epsilon^j q_j(\psi, \theta) \tag{3.2}$$

would break down at some point due to the nonlinearities in (3.1). However, we will show that in fact such a breakdown never occurs; that is, the formal power series is a uniformly valid solution.

Note that if we expand the square root in (3.1) in a power series in ϵ, the equation which results is of the following form:

$$2\frac{\partial q}{\partial \theta} - [\lambda^2 + \epsilon g(q;\epsilon)]\frac{\partial^2 q}{\partial \psi^2} = 0 \qquad 0 \leqslant \theta \leqslant 2\pi,\ \psi \geqslant 0 \tag{3.3}$$

with the added restrictions that $\lambda \neq 0$ (in fact, in our case $\lambda = 1$) and that $g(q;\epsilon)$ has a power-series expansion in q which consists only of positive integral powers.

We postulate the following expansions in ϵ for our functions:

$$f(\theta;\epsilon) = \sum_{j=0}^{\infty} \epsilon^j f_j(\theta) \tag{3.4a}$$

$$g(q;\epsilon) = \sum_{j=0}^{\infty} \epsilon^j g_j(\mathcal{Q}_j) \qquad \text{where } \mathcal{Q}_j = \{q_0, q_1, \ldots, q_j\}. \tag{3.4b}$$

(3.4*b*) follows directly from inserting (3.2) into g. Substituting (3.2) and (3.4) into (3.3) and (2.6), we have a set of equations, one at each integral order of ϵ:

$$2\frac{\partial q_j}{\partial \theta} - \lambda^2 \frac{\partial^2 q_j}{\partial \psi^2} = \mathcal{R}_j = \sum_{k=0}^{j-1} g_k \frac{\partial^2 q_{j-1-k}}{\partial \psi^2} \tag{3.5}$$

$$q_j(\psi, 0) = q_j(\psi, 2\pi) \tag{3.6}$$

$$q_j(\infty, \theta) = 0 \qquad q_j(0, \theta) = f_j(\theta). \tag{3.7}$$

We now wish to prove the following theorem:

Theorem 3.1. *Let* $f(\theta)$ *be piecewise differentiable on* $[0, 2\pi]$ *(that is, a complex Fourier series exists for the function) and slowly varying (that is,* $f'(\theta) = \mathrm{O}(1)$ *for all* $\theta \in [0, 2\pi]$*). Then a solution to (3.3) and (2.6) exists iff* $\bar{f} = 0$*. In addition, the decay rate in* ψ *is no slower than*

$$\exp\left(-\frac{\psi\sqrt{n_0}}{\lambda}\right)$$

where n_0 *is the smallest mode of the regular Fourier expansion of* $f(\theta)$*.*

In essence, theorem 3.1 states that the perturbation expansion is regular with no multiple scales involved. Also, note that the requirement that $\bar{f} = 0$ is equivalent to the requirement that $\overline{U^2} = \overline{F^2}$, which allows us to determine the vorticity of the Euler flow given the boundary conditions. We also note that if we have an $f(\theta)$ whose mean $\bar{f} \neq 0$, we may define $f^*(\theta) = f(\theta) - \bar{f}$, $q^*(\theta) = q(\theta) - \bar{f}$. These new starred quantities then satisfy the hypotheses of the theorem, albeit with a different λ.

Since we know from (2.6*a*) and the statement of the theorem that a Fourier series exists, we can decompose our functions q_j into their complex Fourier coefficients:

$$q_j(\psi, \theta) = \sum_{n=-\infty}^{\infty} q_{j,n}(\psi) \mathrm{e}^{\mathrm{i}n\theta} \qquad q_{j,n}(\psi) = \frac{1}{2\pi}\int_0^{2\pi} q_j(\psi, \theta)\mathrm{e}^{-\mathrm{i}n\theta}\,\mathrm{d}\theta. \tag{3.8}$$

Introducing these substitutions into our system (3.5)–(3.7), we have

$$2\mathrm{i}n q_{j,n} - \lambda^2 q''_{j,n} = \mathcal{R}_{j,n} \tag{3.9}$$

$$q_{j,n}(0) = f_{j,n} \qquad q_{j,n}(\infty) = 0. \tag{3.10}$$

We need several facts in order to prove theorem 3.1. First, we must prove a result about the $\mathcal{R}_{j,0}$:

Lemma 3.2. $\mathcal{R}_{j,0} = 0$ *for all* j.

Proof. Rewriting (3.3) and integrating with respect to θ, we have

$$\int_0^{2\pi} \frac{\partial^2 q}{\partial \psi^2}\, \mathrm{d}\theta = 2\int_{q(\psi,0)}^{q(\psi,2\pi)} \frac{\mathrm{d}q}{\lambda^2 + \epsilon g(q;\epsilon)} = 0 \tag{3.11}$$

where we have used (2.6*a*). By definition, we have that

$$\mathcal{R}_{j,0} = \frac{1}{\pi}\int_0^{2\pi} \frac{\partial q_j}{\partial \theta}\, \mathrm{d}\theta - \frac{\lambda^2}{2\pi}\int_0^{2\pi} \frac{\partial^2 q_j}{\partial \psi^2}\, \mathrm{d}\theta = \frac{1}{\pi}\left[q_j\right]_0^{2\pi} = 0$$

where we have used (3.6) and the fact that (3.11) holds at each order in the expansion of q. □

Corollary 3.3. $q_{j,0} \equiv 0$ *for all* j *iff* $\bar{f} = 0$.

Proof. By the definition of our complex Fourier coefficients, we know that $\bar{f} = 0$ is equivalent to $f_{j,0} = 0$ for all j. By lemma 3.2, we know that $\mathcal{R}_{j,0} = 0$ for all j, so our equation for mode 0 becomes

$$-q''_{j,0} = 0 \qquad q_{j,0}(0) = f_{j,0} \qquad q_{j,0}(\infty) = 0.$$

Due to the nature of the operator and the boundary condition at infinity, this equation has a solution if and only if $f_{j,0} = 0$. □

The physical reasoning behind corollary 3.3 is the following. If $\bar{f} \neq 0$, we cannot satisfy our far-field condition that q must decay to 0. Therefore, we must not be using the proper U when constructing our displacement field q. This yields the consistency condition that $\overline{U^2} = \overline{F^2}$. Therefore, we see that we may bypass the particulars of the viscous boundary layer if all we wish to calculate is the Euler flow. This approach was taken by Batchelor [1].

Next we will need a fact regarding the products of eigenfunctions of the operator in (3.9).

Corollary 3.4. *Let* $\phi_n(\psi)$ *be the* n*th eigenfunction of the operator in (3.9) and*

$$\Phi_{n,\{(a_j,b_j)\}}(\psi) \equiv \prod_{j=1}^{m} \phi_{a_j}^{b_j}(\psi) \qquad \text{where } \sum_{j=1}^{m} a_j b_j = n.$$

Then

$$\Phi_{n,\{(a_j,b_j)\}}(\psi) = \phi_n(\psi)$$

iff $m-1$ *of the* a_j *are* 0, *and* $b_J = 1$ *for the one* $a_J \neq 0$.

Proof. First we calculate the necessary eigenfunctions. They are

$$\phi_n(\psi) = \exp\left(-\frac{\psi\sqrt{2\mathrm{i}n}}{\lambda}\right) = \exp\left\{-[1 + (\operatorname{sgn} n)\mathrm{i}]\frac{\psi\sqrt{|n|}}{\lambda}\right\}. \tag{3.12}$$

Then we have that

$$\left|\prod_{j=1}^{m} \phi_{a_j}^{b_j}\right| = \left|\prod_{j=1}^{m} \exp\left\{-b_j[1 + (\operatorname{sgn} a_j)\mathrm{i}]\frac{\psi\sqrt{|a_j|}}{\lambda}\right\}\right| = \exp\left(-\sum_{j=1}^{m} b_j \frac{\psi\sqrt{|a_j|}}{\lambda}\right).$$

But from the triangle inequality we have that

$$\sum_{j=1}^{m} b_j |a_j|^{1/2} \geqslant \left|\sum_{j=1}^{m} a_j b_j\right|^{1/2}$$

with equality iff $m-1$ of the a_j are 0, and $b_j = 1$ if $a_j \neq 0$. Therefore, we have

$$\left|\Phi_{n,\{(a_j,b_j)\}}(\psi)\right| \equiv \left|\prod_{j=1}^{m} \phi_{a_j}^{b_j}\right| \leqslant \exp\left(-\frac{\psi}{\lambda}\left|\sum_{j=1}^{m} a_j b_j\right|^{1/2}\right) = \exp\left(-\frac{\psi\sqrt{|n|}}{\lambda}\right) \equiv |\phi_n(\psi)|$$

with equality iff $m-1$ of the a_j are 0 and $b_J = 1$ for the one $a_J \neq 0$. In this case, we see that $\Phi_{n,\{(a_j,b_j)\}}(\psi) = \phi_{a_J}(\psi)$, and hence the corollary is proved. □

Remarks.

(1) $\Phi_{m,\cdot}\Phi_{n-m,\cdot} = \Phi_{n,\cdot}$.

(2) $\Phi''_{n,\cdot}$ is proportional to $\Phi_{n,\cdot}$.

(3) If a term of the form $e^{\alpha\psi}$ appears in $\mathcal{R}_{j,n}$, then a corresponding term

$$\frac{e^{\alpha\psi}}{2in - \alpha^2}$$

must appear in $q_{j,n}$ if $\alpha^2 \neq 2in$. Note that the case $\alpha^2 = 2in$ causes a secularity.

We need only one more fact to complete the framework of our proof.

Lemma 3.5. *For all j and n,*

(a) $q_{j,n}(\psi)$ is a sum of terms, each of which is proportional to one of the $\Phi_{n,\cdot}$.

(b) $g_{j,n}(\psi)$ is a sum of terms, each of which is proportional to one of the $\Phi_{n,\cdot}$.

(c) $\mathcal{R}_{j,n}(\psi)$ is a sum of terms, each of which is proportional to one of the $\Phi_{n,\cdot}$, but does not contain any terms of the form $\phi_n(\psi)$.

Proof. First we compute the $\mathcal{R}_{j,n}$:

$$\mathcal{R}_{j,n} = \left(\sum_{k=0}^{j-1} g_k q''_{j-1-k}\right)_{,n} = \sum_{k=0}^{j-1}\sum_{m=-\infty}^{\infty} g_{k,m} q''_{j-k-1,n-m}. \tag{3.13}$$

We induct on j. Since $\mathcal{R}_{0,n} = 0$, the solution to (3.9) and (3.10) at leading order is given by

$$q_{0,n}(\psi) = f_{0,n}\phi_n(\psi) \tag{3.14}$$

which satisfies all of the hypotheses of our theorem. Now given that the lemma is true for j, we show that it is true for $j+1$. We first note that $\Phi_{n,\cdot}$ is in the form $e^{\alpha\psi}$, so by remark (3) we know that since $\Phi_{n,\cdot} \neq \phi_n$, if there exists a term of that form in $\mathcal{R}_{j,n}$, there must exist a corresponding term in $q_{j,n}$.

From the form of (3.13) we see that first we must show that $g_{k,m}$ has a power series expansion in the $\Phi_{m,\cdot}$. But since $g_{k,m}$ has a power series in the set $\mathcal{Q}_k$, and we have from our induction hypothesis that (a) is true for all $k < j+1$, then we see that $g_{k,m}$ must also have a power series in the $\Phi_{m,\cdot}$. Therefore, our proof of part (b) is complete.

In addition, by remark (2) $q''_{j-k,n-m}$ consists of a sum of terms proportional to the $\Phi_{n-m,\cdot}$. Therefore, by (3.13) we see that each $\mathcal{R}_{j+1,n}$ is made up of a sum of pairwise products of terms proportional to $\Phi_{m,\cdot}$ and $\Phi_{n-m,\cdot}$. But by remark (1) each of these terms is proportional to $\Phi_{n,\cdot}$. In addition, by corollary 3.4, none of these terms can be proportional to ϕ_n unless many of the terms are proportional to ϕ_0. But by corollary 3.3, $q_{j,0} = 0$ for all j, so there can *never* be a term of the form ϕ_0. So our proof of part (c) is complete.

From remark (3) we note that if $\mathcal{R}_{j+1,n}$ is in the form of a sum of terms proportional to $\Phi_{n,\cdot}$, then $q_{j+1,n}$ must be of that same form, with the exception that in order to satisfy the boundary condition, we may have to add a term of the form ϕ_n. So our proof of part (a) is also complete. □

Therefore, we now have the tools necessary to complete the proof of theorem 3.1.

Proof of theorem 3.1. The fact that $f(\theta)$ is slowly varying allows us to use (3.5)–(3.7) as a system to model (3.3) and (2.6) without worrying about interior layers in θ. (The case where there are discontinuities in the wall data will be considered in the next section.) The fact that $f(\theta)$ has a complex Fourier series allows us to use (3.9) and (3.10).

If $\bar{f} = 0$, then corollary 3.3 holds, which means that lemma 3.5 is true. Therefore, a solution exists with no secularity. What is the decay rate of the solution? To answer this, we simply solve (3.9) and (3.10) to leading order. By equations (3.8), (3.12), and (3.14), we have that our solution is

$$q(\psi, \theta; \epsilon) = \sum_{n=-\infty}^{\infty} f_{0,n} \exp\left\{-[1 + (\operatorname{sgn} n)\mathrm{i}]\frac{\psi\sqrt{|n|}}{\lambda} + \mathrm{i}n\theta\right\} + \mathrm{O}(\epsilon).$$

Let n_0 be the smallest mode (in absolute value) for which $f_{0,n} \neq 0$, so n_0 is the smallest mode in the Fourier expansion of $f(\theta)$. Then it is easy to see that any eigenfunctions with $|n| > n_0$ decay at a faster rate, and so we have

$$|q(\psi, \theta; \epsilon)| \sim 2\Re f_{0,n_0} \exp\left(-\frac{\psi\sqrt{n_0}}{\lambda}\right) \qquad \psi \to \infty$$

which completes our proof. □

Therefore, from theorem 3.1 we see that even though (3.1) is a nonlinear equation, we obtain the rather surprising result that a regular perturbation expansion converges to the correct solution. Hence, our expansion depends on the displacement scale ϵ only as a gauge function, and our solution at each order depends only on the viscous stream scale. Though theorem 3.1 is an asymptotic result which holds in the limit that $\epsilon \to 0$, this result has been extended to show convergence in the case of finite perturbations by Kim [4].

4. Discontinuous boundary data

We now consider the case where there are discontinuities in $f(\theta)$. Such discontinuities could come about in the case of an arc-shaped sleeve rotating at a slightly different speed from the cylinder. A variant of this situation is discussed in Batchelor [1]. Such a case is not covered by the hypotheses of theorem 3.1 because our functions are no longer smooth. However, we know from Fourier theory that if there is a discontinuity in $f(\theta)$ at $\theta = \theta_d$, then the Fourier series for $f(\theta)$ will converge to

$$\frac{f(\theta_d^+) + f(\theta_d^-)}{2} \tag{4.1}$$

at θ_d. Is this also the solution of the physical system? If so, then theorem 3.1 will hold even in the case where $f(\theta)$ is not continuous.

We introduce the following scaled variables:

$$\tilde{\theta} = \frac{\theta - \theta_d}{\epsilon^2} \qquad \tilde{\psi} = \frac{\psi}{\epsilon} \qquad q(\psi, \theta) = w(\tilde{\psi}, \tilde{\theta}) - \frac{f(\theta_d^+) + f(\theta_d^-)}{2}. \tag{4.2}$$

Using (4.2) in (3.1) and (2.6*b*), we have the following system:

$$2\frac{\partial w}{\partial \tilde{\theta}} - \sqrt{1 + \epsilon w}\frac{\partial^2 w}{\partial \tilde{\psi}^2} = 0 \qquad -\infty < \tilde{\theta} < \infty,\ \tilde{\psi} > 0 \tag{4.3a}$$

$$w(\tilde{\psi}, \pm\infty) = \pm\left[\frac{f(\theta_d^+) - f(\theta_d^-)}{2}\right] \qquad w(0, \tilde{\theta}) = \operatorname{sgn}(\tilde{\theta})\left[\frac{f(\theta_d^+) - f(\theta_d^-)}{2}\right]. \tag{4.3b}$$

We begin by constructing the leading-order solution by letting $w(\tilde{\psi}, \tilde{\theta}) \sim w_0(\tilde{\psi}, \tilde{\theta}) + o(1)$. Noting that a similarity solution is expedient, we introduce the following variables:

$$\zeta = \frac{\tilde{\psi}}{(2|\tilde{\theta}|)^{1/2}} \qquad w_0(\tilde{\psi}, \tilde{\theta}) = \begin{cases} w_0^+(\zeta) & \tilde{\theta} > 0 \\ w_0^-(\zeta) & \tilde{\theta} < 0. \end{cases}$$

Making these substitutions in (4.3), we have

$$\frac{d^2 w_0^\pm}{d\zeta^2} + 2\zeta \frac{dw_0^\pm}{d\zeta} = 0 \qquad \zeta > 0 \tag{4.4a}$$

$$w_0^\pm(0) = \pm\left[\frac{f_0(\theta_d^+) - f_0(\theta_d^-)}{2}\right] \qquad w_0^\pm(\infty) = 0 \tag{4.4b}$$

the solution of which is

$$\begin{aligned} w_0^\pm(\zeta) &= \pm\left[\frac{f_0(\theta_d^+) - f_0(\theta_d^-)}{2}\right] \operatorname{erfc} \zeta \\ w_0(\tilde{\psi}, \tilde{\theta}) &= \operatorname{sgn}(\tilde{\theta}) \left[\frac{f_0(\theta_d^+) - f_0(\theta_d^-)}{2}\right] \operatorname{erfc}\left(\frac{\tilde{\psi}}{(2|\tilde{\theta}|)^{1/2}}\right). \end{aligned} \tag{4.5}$$

Since all the derivatives of $w_0^\pm$ with respect to $\tilde{\theta}$ vanish at $\tilde{\theta} = 0$, we see that our solution in (4.5) is as smooth as necessary.

In a similar manner, one can show that at any order the system equilibrates to the one given by the Fourier series. Therefore, we see that there is an agreement between the mathematical and physical solutions of the problem. Any discontinuities in our imposed wall conditions are quickly smoothed on a much faster stream scale than the rest of the viscous flow. This smoothed flow satisfies the hypotheses of theorem 3.1, and hence we know that a smooth flow exists for the entire viscous boundary layer.

We note that due to the relationship between $\tilde{\psi}$ and $\tilde{\theta}$, our expression can be written as

$$w_0\left(\frac{\psi}{\epsilon}, \frac{\theta - \theta_d}{\epsilon^2}\right) = \operatorname{sgn}(\theta - \theta_d) \left[\frac{f_0(\theta_d^+) - f_0(\theta_d^-)}{2}\right] \operatorname{erfc}\left(\frac{\psi}{(2|\theta - \theta_d|)^{1/2}}\right). \tag{4.6}$$

Thus, we see that ϵ scales out of our problem, as one would expect since we are using similarity variables. Since there is no pressure gradient, we see that our solution at each order is dependent on the displacement scale only as a gauge function, as found in section 3.

Since a specific function of ϵ does not appear in our interior layer solution, we would expect that even for ϵ moderate but still $o(1)$, we could scale by some small function of ϵ in order to obtain (4.6). On the other hand, if the discontinuties were $O(1)$ quantities, they might induce multiple eddies and other effects which we do not consider in this work. A brief discussion of such a system may be found in Batchelor [1].

5. Including a pressure gradient

Now we wish to consider the case where a pressure gradient exists: by (2.3), we see that $U(\theta)$ is not constant. Equations of the form of (2.5) with nonconstant U can also arise, after a suitable change of variable, from imposing uniform boundary conditions on slightly perturbed cylinders [6]. We note from theorem 3.1 that the ϵq term in the square root will not cause any secularities, since the dying exponentials will multiply in such a way that the right-hand side does not contain any eigenfunctions. However, we also have θ-dependent terms in our square root. How will these new terms affect our solution?

We begin by letting $U^2(\theta) = 1 + \epsilon h(\theta; \epsilon)$, where

$$h(\theta; \epsilon) = \sum_{j=0}^{\infty} \epsilon^j h_j(\theta). \tag{5.1}$$

Therefore, (2.5) can be written in the form (analogous to (3.3))

$$2\frac{\partial q}{\partial \theta} - [1 + \epsilon g(q, \theta; \epsilon)]\frac{\partial^2 q}{\partial \psi^2} = 0 \qquad 0 \leqslant \theta \leqslant 2\pi, \ \psi \geqslant 0 \tag{5.2}$$

where $g(q, \theta; \epsilon)$ is no longer arbitrary, but is given by expanding the square root in (2.5). Now we are prepared to prove the following theorem.

Theorem 5.1. *Let $f(\theta)$ and $h(\theta)$ be piecewise differentiable on $[0, 2\pi]$ (that is, a complex Fourier series exists for each function) and slowly varying (that is, $f'(\theta)$ and $h'(\theta) = \mathrm{O}(1)$ for all $\theta \in [0, 2\pi]$). Then a solution of (5.2) and (2.6) exists iff the system is consistent. Furthermore, the consistency conditions to $\mathrm{O}(\epsilon)$ are*

$$\bar{f}_0 = 0 \qquad \overline{h_0 f_0} = 2\bar{f}_1. \tag{5.3}$$

This consistency relationship between the mean of terms in the expansion of U and the mean of terms in the expansion of q was noted by Wood [6] and Feynman and Lagerstrom [3]. As remarked earlier, these consistency conditions determine U (since f is a known quantity).

Proof. The fact that $f(\theta)$ and $h(\theta)$ are slowly varying allows us to use (5.2) and (2.6) directly without the use of interior layers like those in section 4. We begin by writing the first three equations in our perturbation expansion obtained by using (3.2) and (3.4):

$$2\frac{\partial q_0}{\partial \theta} - \frac{\partial^2 q_0}{\partial \psi^2} = 0 \tag{5.4a}$$

$$q_0(0, \theta) = f_0(\theta) \qquad q_0(\infty, \theta) = 0 \tag{5.4b}$$

$$2\frac{\partial q_1}{\partial \theta} - \frac{\partial^2 q_1}{\partial \psi^2} = \frac{h_0}{2}\frac{\partial^2 q_0}{\partial \psi^2} + \mathcal{R}_1 \tag{5.5a}$$

$$q_1(0, \theta) = f_1(\theta) \qquad q_1(\infty, \theta) = 0 \tag{5.5b}$$

$$2\frac{\partial q_2}{\partial \theta} - \frac{\partial^2 q_2}{\partial \psi^2} = \frac{h_1}{2}\frac{\partial^2 q_0}{\partial \psi^2} + \frac{h_0}{2}\frac{\partial^2 q_1}{\partial \psi^2} - \frac{h_0^2}{8}\frac{\partial^2 q_0}{\partial \psi^2} + \mathcal{R}_2 \tag{5.6a}$$

$$q_2(0, \theta) = f_2(\theta) \qquad q_2(\infty, \theta) = 0 \tag{5.6b}$$

where the $\mathcal{R}_i$ notation is the same as in section 3.

Using the Fourier series expansion, we have that $q_{0,n}$ is once again given by (3.14), and exists if and only if $\bar{f}_0 = 0$. Therefore, we have the first condition in (5.3). In addition, in Fourier space equations (5.5) become

$$2\mathrm{i}nq_{1,n} - q''_{1,n} = \mathrm{i}\sum_{k=-\infty}^{\infty} kh_{0,n-k} f_{0,k}\phi_k(\psi) + \mathcal{R}_{1,n} \tag{5.7a}$$

$$q_{1,n}(0) = f_{1,n} \qquad q_{1,n}(\infty) = 0. \tag{5.7b}$$

For any $n \neq 0$, we see that we may add a multiple of an eigenfunction to satisfy our boundary condition at $\psi = 0$. However, for $n = 0$, our eigenfunctions do not satisfy the matching condition as $\psi \to \infty$. Therefore, for a solution to exist, our far-field and boundary conditions must be consistent. We know from lemma 3.2 that $\mathcal{R}_{1,0} = 0$. Therefore, solving (5.7) for the mode $n = 0$, we have

$$q_{1,0} = \tfrac{1}{2}\sum_{k=-\infty}^{\infty} h_{0,-k} f_{0,k}\phi_k(\psi) \qquad \bar{f}_1 = \frac{1}{2}\sum_{k=-\infty}^{\infty} h_{0,-k} f_{0,k} = \frac{\overline{h_0 f_0}}{2}.$$

Thus, the second equation in (5.3) follows, and the theorem is proved. □

Remark. To $O(\epsilon)$, equations (5.3) replace the requirement in section 3 that $\bar{f} = 0$. These conditions are expressed in a different form by Wood [6] and Feynman and Lagerstrom [3]. However, note that in the case $h_0 \equiv 0$ (which to leading order is the case without a pressure gradient), equations (5.3) reduce to $\bar{f} = 0$ to leading two orders.

Since we have assumed that we have normalized by the proper value for the *inviscid* flow, we have that $\bar{h}_j = 0$ for all j. We continue our analysis by checking the form of the secularity that will arise.

Theorem 5.2. *Let $f(\theta)$ and $h(\theta)$ satisfy the hypotheses of theorem 5.1. In addition, let $\bar{h} = 0$. Then the solution of (5.2) and (2.6) can be expressed as a regular perturbation expansion of a function of two variables:*

$$q(\psi, \theta; \epsilon) = \sum_{j=0}^{\infty} \epsilon^j Q_j(\hat{\psi}, \Psi, \theta) \qquad \hat{\psi} = \epsilon^2\psi \qquad \Psi = \psi\left(1 + \sum_{j=3}^{\infty} \omega_j \epsilon^j\right) \tag{5.8}$$

whenever $h_0(\theta) \not\equiv 0$.

Proof. We begin by examining equation (5.7*a*). Since $\bar{h} = 0$, we know that $h_{0,0} = 0$. Therefore, we immediately see from equation (5.7*a*) that an eigenfunction does not appear in that term in the right-hand side. We know from theorem 3.1 that the $\mathcal{R}_i$ do not contribute to secularities, so we have that at this order there is no secularity present.

Now we continue our solution to next order by looking at only the secular terms. We see that since $\bar{h}_0 = 0$, the middle term on the right-hand side of (5.6*a*) will not cause secularities. In addition, since $h_{1,0} = 0$, we know that no secularity can arise from the first term on the right-hand side of (5.6*a*). However, we see that the third term will cause a secularity whenever $h_0 \not\equiv 0$, for then we have in Fourier space

$$2\mathrm{i}nq_{2,n} - q''_{2,n} = -\frac{\mathrm{i}nf_{0,n}}{4}\overline{h_0^2}\phi_n(\psi) + \cdots$$

the solution of which is

$$q_{2,n}(\psi) = -\frac{f_{0,n}\psi\overline{h_0^2}}{8}\sqrt{\frac{\mathrm{i}n}{2}}\phi_n(\psi) + \cdots \tag{5.9}$$

where the unlisted terms are not related to a secularity. The form of (5.9) motivates the choice of variables in (5.8).

One could argue that a term of the form in (5.9) is not problematic, since it still decays as $\psi \to \infty$. However, the form of (3.2) is based upon the assumption that $q_{i+1} = o(q_i)$ as $\epsilon \to 0$ *uniformly* in our domain. It is clear from (5.9) that this is not the case for $\psi = O(\epsilon^{-2})$. Therefore, though both terms decay, we note that there must be decay or oscillation occurring on a longer scale in ψ (in this case, $\epsilon^2\psi$). Hence, a multiple-scale expansion is necessary for a uniformly valid solution.

Using (5.8) and the Fourier series approach, we see that our leading-order solution, analogous to (3.14), is given by

$$Q_{0,n}(\hat{\psi}, \Psi) = c_{0,n}(\hat{\psi})\phi_n(\Psi) \qquad c_{0,n}(0) = f_{0,n}. \tag{5.10}$$

In addition, the equation analogous to (5.7*a*) becomes

$$2\mathrm{i}nQ_{1,n} - \frac{\partial^2 Q_{1,n}}{\partial \Psi^2} = \mathrm{i}\sum_{k=-\infty}^{\infty} kh_{0,n-k}c_{0,k}(\hat{\psi})\phi_k(\Psi) + \mathcal{R}_{1,n}.$$

in Fourier space. We see that since no secularity occurs at this order, we did not include an ω_1 term in our definition of Ψ; neither did we let our slow scale $\hat{\psi} = \epsilon\psi$.

The $\mathrm{O}(\epsilon^2)$ equation is given by

$$2\frac{\partial Q_2}{\partial \theta} - \frac{\partial^2 Q_2}{\partial \Psi^2} = \frac{h_1}{2}\frac{\partial^2 Q_0}{\partial \Psi^2} + \frac{h_0}{2}\frac{\partial^2 Q_1}{\partial \Psi^2} - \frac{h_0^2}{8}\frac{\partial^2 Q_0}{\partial \Psi^2} + 2\frac{\partial^2 Q_0}{\partial \Psi \partial \hat{\psi}} + \mathcal{R}_2$$

which in Fourier space becomes

$$2\mathrm{i}nQ_{2,n} - \frac{\partial^2 Q_{2,n}}{\partial \Psi^2} = -\frac{\mathrm{i}nc_{0,n}(\hat{\psi})}{4}\overline{h_0^2}\phi_n(\Psi) - 2c'_{0,n}(\hat{\psi})\sqrt{2\mathrm{i}n}\phi_n(\Psi) + \cdots \tag{5.11}$$

where the unlisted terms do not contribute to secularity. Suppressing the secular-causing terms in (5.11), we have

$$c_{0,n}(\hat{\psi}) = f_{0,n}\exp\left(-\frac{\overline{h_0^2}\hat{\psi}}{16}\sqrt{2\mathrm{i}n}\right) = f_{0,n}\phi_n(\overline{h_0^2}\hat{\psi}/16) \tag{5.12}$$

where we have used the initial conditions in (5.10). Hence our proof is complete. □

Remarks.

(1) The first two terms of the Taylor expansion of (5.12) for small $\hat{\psi}$ agree with our results in (3.14) and (5.9).

(2) Since $c_{0,0}$ is of the same form as ϕ_0 (and hence does not decay as $\hat{\psi} \to \infty$), we see that introducing the new variable $\hat{\psi}$ does nothing to affect our consistency conditions (5.3).

(3) Our hypothesis $\bar{h} = 0$ assumes that the means of the wall velocity and the inviscid velocity are nearly the same and that this mean can be used as a characteristic velocity with which to normalize. If this assumption is relaxed, the multiple-scale procedure still works. However, the slow-stream scale must be $\hat{\psi} = \epsilon\psi$.

Therefore, we see that we have both an amplitude and a phase modulation of our solution coming from the slow stream scale $\hat{\psi}$. We also note that both the amplitude and phase of our solution will depend on *all* integral orders of ϵ smaller than ϵ, something which did not occur when there was no pressure gradient. Thus, we see that in this case the displacement scale does play a role in the solution at each order, providing the necessary scaling for the slow-stream variable. This is more in keeping with what one would expect in a nonlinear problem.

The framework outlined in this section is quite versatile since it does not depend on the relative size of our parameters Re^{-1} and ϵ. It is our hope that this framework can be formally extended to more difficult problems, such as the cylindrical analogue of a Hills spherical vortex [2] or the bifurcation of a small inviscid eddy from the wall.

6. Conclusions

Since the dynamics of flow with closed streamlines are so complicated, researchers have focused on specific facets of the flow to understand better the flow as a whole. Batchelor [1] was interested in obtaining the Bernoulli constant of the system. Wood [6] made some cursory inspections of the flow in the boundary layer, but did not thoroughly examine the physical ramifications of his results.

In this work, we have carefully examined the flow in a viscous boundary layer and how such a flow might have far-reaching effects in the inviscid region. Though we modelled the case where the wall velocity was perturbed, the form of (2.5) could also come from a uniform boundary condition on a perturbed cylindrical shape, another case of interest [6].

By far the simplest of these problems is that of a circular cylinder, where the vorticity is known *a priori* in terms of the wall data, at least whenever the boundary layer exists. For this case, we proved that a regular perturbation expansion suffices to describe the flow in the viscous boundary layer. The decay rate of the flow with respect to the stream function was found to depend on the Fourier modes of the perturbed wall velocity. No multiple-scale expansion was necessary, and hence we saw that the solution depended on the amplitude of the disturbance only through gauge functions.

In the case where there was a small discontinuity in our imposed wall velocity, we saw that the physical system quickly smoothed out the discontinuity so that our Fourier series solution still held. Note that this was true only when the discontinuities were in small deviations from the flow's constant value.

In the case where there was a pressure gradient, the dynamics of the flow changed dramatically. Even though the pressure gradient was caused by only slight perturbations from the case without a pressure gradient, our regular perturbation expansion was no longer uniformly valid. We calculated the needed variables for our expansion, and proved that a multiple-scale expansion is always needed if there is a pressure gradient. The slow-stream scale needed reflected the amplitude of the displacement. In addition, we replicated the result shown in Feynman and Lagerstrom [3] indicating consistency conditions that $U(\theta)$ and $f(\theta)$ must satisfy in order for our problem to have a solution.

If $\epsilon = O(1)$, the present theory becomes invalid. Since we make no assumption about the relative size of our parameters, we see that in this limit, two different types of behaviour can occur. If the Reynolds number is bounded from above by ϵ^{-1}, then we see that a finite disturbance corresponds to a finite Reynolds number. Therefore, these disturbances would then be spread throughout the cylinder, since our Euler assumption would no longer be valid. However, if the Reynolds number is bounded from below by ϵ^{-1}, we see that we can still have an Euler flow with a finite disturbance.

In physical terms, our theory becomes invalid with the occurrence of separation. This can occur through the combined influence of the pressure gradient and the wall data. It can take several forms, but the most interesting in the present context is *marginal separation*. The onset of marginal separation is marked by the detachment of the wall streamline and its immediate reattachment, forming a small boundary-layer eddy. As a result of the multiscaling procedure in the case where $0 < \mathrm{Re}^{-1} \ll \epsilon \ll 1$, we have a new length scale which depends on ϵ and is larger than the boundary layer thickness. As ϵ increases or the wall data is further modified, the eddy can grow and change the topology of the Euler flow to a two-eddy configuration. It is in this way that various eddy configurations and their relation to wall geometry and data can be explored.

Of course, the present theory does not consider the stability of these flows. Solid-body rotation is a stable configuration, so we are dealing with weakly unstable flows. At large but finite Reynolds numbers (around 500), stable flows of this kind can be observed and a viscous correction to the Euler flow can be calculated [4]. Therefore, it is our hope that the mathematical framework presented in this paper can be used to analyse more difficult problems.

7. Nomenclature

7.1. *Variables and parameters*

The equation number where a particular quantity first appears is listed, if appropriate.

a	arbitrary constant.
b	arbitrary constant.
$c(\hat{\psi})$	coefficient of terms in two-variable expansion (5.10).
$F(\theta)$	imposed wall velocity $u(0,\theta)$ (2.2).
$f(\theta)$	imposed wall displacement velocity $q(0,\theta)$ (2.6*b*).
$g(\cdot;\epsilon)$	power-series expansion of nonlinear terms in equation (2.5), (3.3).
$h(\theta)$	deviation of $U(\theta)$ from a constant value, defined as $[U^2(\theta)-1]/\epsilon$ (5.1).
j	indexing variable (3.2).
k	indexing variable (3.5).
m	arbitrary constant.
n	indexing variable (3.8).
$p(\theta)$	pressure of fluid in a cylinder (2.3).
$\mathcal{Q}$	a set of terms in the perturbation expansion of q (3.4*b*).
$Q(\hat{\psi},\Psi,\theta)$	element of perturbation expansion for q in multiple-scale expansion procedure (5.8).
$q(\cdot)$	velocity displacement field, defined as $(u^2-U^2)/\epsilon$ (2.5).
$\mathcal{R}$	the right-hand side of an equation in our perturbation expansion (3.5).
Re	Reynolds number of the system.
$U(\theta)$	Euler flow velocity near wall (2.1*a*).
$u(\cdot,\theta)$	angular velocity of fluid in boundary layer (2.1*a*).
$v(\cdot,\theta)$	scaled radial velocity in boundary layer (2.1*a*).
$w(\tilde{\psi},\tilde{\theta})$	velocity displacement in boundary layer near wall velocity discontinuity (4.2).
y	spatial variable in viscous boundary layer (2.1*a*).
$\mathcal{Z}$	the integers.
α	arbitrary exponent.
ϵ	perturbation parameter.
ζ	similarity variable in boundary layer near wall velocity discontinuity.
θ	cylindrical angular coordinate (2.1*a*).
λ	first term in the expansion of the nonlinear terms in (3.1*a*), (3.3).
$\Phi(\psi)$	a product of eigenfunctions ϕ.
$\phi(\cdot)$	an eigenfunction of the operator in (3.9).
Ψ	perturbed stream function (5.8).
ψ	stream function.
ω	coefficient in perturbation expansion for Ψ (5.8).

7.2. *Other notation*

d	as a subscript, used to indicate a position in θ at which the imposed wall velocity is discontinuous (4.1).
J	as a subscript, used to indicate that j such that $a_j \neq 0$.
$n \in \mathcal{Z}$	as a subscript, used to indicate a term in an expansion, either in ϵ (3.2) or Fourier mode (3.8).

$*$	as a superscript, used to indicate a variable which satisfies the hypotheses of theorem 3.1.
$+$	as a superscript on w_0, used to indicate the solution for $\zeta > 0$.
$-$	as a superscript on w_0, used to indicate the solution for $\zeta < 0$.
$\Re$	the real part of an expression.
$\bar{}$	used to indicate average values.
$\hat{}$	used to indicate the slow-stream variable (5.8).
$\tilde{}$	used to indicate scaled coordinates near a discontinuity in the boundary condition (4.2).

Acknowledgments

The author thanks Professor W Stephen Childress and Dr S-C Kim of the Courant Institute for comments on this research. This work was performed under National Science Foundation grant NSF-9407531. Many of the calculations herein were performed using Maple.

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