

An Application of Matrix Theory to the Evolution of Coupled Modes*

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Abstract. In order to overcome loss in optical fibers, experimentalists are interested in employing parametric amplifiers using four-wave mixing. Upon linearizing the nonlinear Schrödinger equation typically used as a model for such amplifiers, a system of ODEs results for the complex amplitude. The solution can also be expressed as the product of transfer matrices and the initial condition and its conjugate. Physical insight about the fiber-optic system can be gained by examining the theoretical properties of the matrices in the mathematical system. This module, suitable for inclusion in an advanced undergraduate or graduate linear algebra course, explores these properties and should provide a good physical motivation for the theoretical explorations in such a course.

Key words. education, eigenvalues, fiber optics, matrix theory, singular value decomposition

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I. Introductions.

I.1. To the Instructor. Most textbooks on “linear algebra with applications” confine their discussion of applications to computational examples. Hence when teaching a course in linear algebra, it is often difficult to find applications of the theoretical aspects of the course. During the 27th Annual Workshop on Mathematical Problems in Industry (MPI) [33] held at the New Jersey Institute of Technology, the coauthors studied a problem related to photon generation in optical fibers. The resulting mathematical problem led to a careful theoretical study of certain matrices modeling the physical system. The work was then abridged to form this module; it discusses theoretical properties of matrices, addressing the specific issues arising in photon generation.

This module is designed to be used by students who are familiar with material up to and including the matrix exponential, as well as the properties of symmetric matrices.

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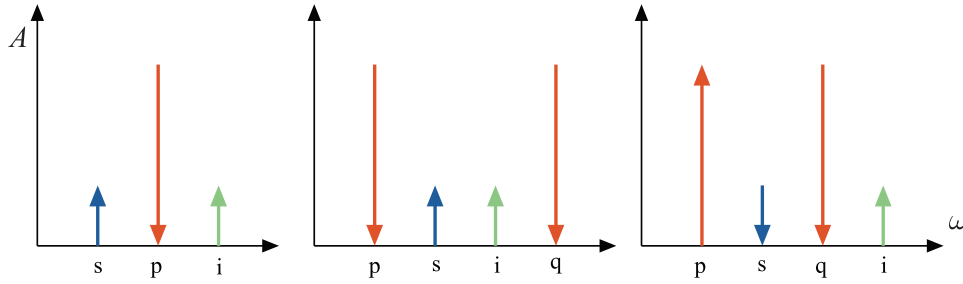


Fig. 1.1 Different types of FWM. Long arrows denote strong pump waves (p and q), and short arrows denote weak signal and idler waves (s and i). Downward arrows denote modes that lose photons, whereas upward arrows denote modes that gain photons. The directions of the arrows are reversible. Left: Modulational instability (degenerate FWM), in which a single pump field (p) transfers photons to signal (s) and idler (i) sidebands. Center: Phase conjugation, in which a phase-conjugated copy of the signal is transferred to the idler at a similar frequency. Right: Frequency conversion, in which a copy of the signal is transferred to the idler at a different frequency.

ces. The module covers such advanced topics as complex, Hermitian, and symplectic matrices, as well as the singular value decomposition (SVD). All of these concepts arose directly out of questions posed in the workshop. Necessary properties are either explained or referenced within.

1.2. To the Student. The two most celebrated developments leading to today's high-speed optical communication networks are the invention of the laser in 1960 [37] and the subsequent decade of development of extremely transparent fused silica for use in optical fiber [38]. Although optical fibers transmit light efficiently, their response to the incident electric fields is nonlinear, meaning that copropagating light waves are affected by each other's presence. This effect is sometimes called the *optical Kerr nonlinearity* [1, section 1.3], [3, section 4.1]. With careful engineering of the optical fiber and related equipment, this nonlinearity can counterbalance chromatic dispersion, the phenomenon whereby waves with different wavelengths (colors) travel with different speeds. Hence the optical fiber system can exhibit robust pulses that are ideal carriers of digital information. The nonlinear optical susceptibility (material response) of fiber also provides a means to transfer energy between different frequencies of light, a mechanism that can be exploited for signal amplification at high powers or for quantum information experiments at extremely low powers.

Transmitting these fields over long distances necessarily involves loss of signal strength due to absorption and scattering of light by the fiber; hence amplification is necessary. Most currently available amplifiers are *phase-insensitive*, so they produce signal gain that is independent of the phase of the complex-valued signal. However, *phase-sensitive* amplifiers are desirable due to their noise reduction and other properties [13], [17], [19], [35], [44]. One novel amplification device uses parametric devices based on four-wave mixing (FWM) [10], [28], [41]. Such devices can phase-conjugate, regenerate, and sample optical signals in classical communication systems [29] (see Figure 1.1). They can also generate photon pairs for quantum information (communication and computation) experiments [8]. As shown in Figure 1.2, these devices can also convert the frequency of the optical signals using the mechanism on the right of Figure 1.1 [29].

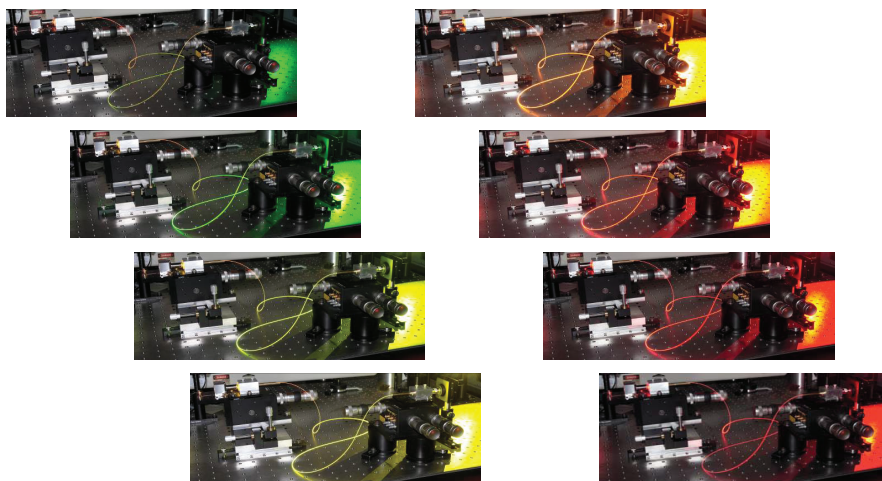


Fig. 1.2 Experiment illustrating modulation instability [36], [43]. In this experiment, the pump wavelength was varied by only a few nm (relative to the zero-dispersion wavelength of the fiber) between the first and last pictures. Because of the dispersion properties of the fiber, the wavelengths at which the signal and idler radiation were generated varied by hundreds of nm, thus changing the color of the emitted light.

Light-wave propagation in a fiber is governed by the nonlinear Schrödinger equation (NLSE) [11], [12], a partial differential equation (PDE) that describes how the complex vector-valued temporal envelope of an electric field evolves as it travels along an optical fiber. The NLSE is suitable for describing pulsed light with time durations down to several picoseconds, as well as continuous-wave (cw) light with a complex amplitude that is nearly constant in time. Many applications, from amplification in optical communications networks to photon generation in quantum experiments, involve one or more large-amplitude *pump fields* interacting with one or more small-amplitude *signal fields*. This scale disparity allows the PDE to be linearized about the equilibrium solution for a pump field with constant power, reducing the complexity of the mathematical equations to be solved. In this case, the cw fields have constant wavelength and amplitudes that depend only on z , the distance along the fiber axis.

In particular, parametric interactions of weak sidebands driven by strong pumps are governed by coupled-mode equations (CMEs) of the following form:

$$(1.1a) \quad \frac{d\mathbf{x}}{dz} = A\mathbf{x} + B\bar{\mathbf{x}},$$

where in uniform media the coupling coefficients that form the entries of $A, B \in \mathbb{C}^{n \times n}$ are constant [20], [30], [31], and \mathbf{x} and $\bar{\mathbf{x}}$ are complex conjugate vectors. (A full list of the variables used in this article is given in Table 1.1.)

Recall that a complex scalar $x \in \mathbb{C}$ can be written $x = \text{Re}\{x\} + i \text{Im}\{x\}$ with *real part* $\text{Re}\{x\}$, *imaginary part* $\text{Im}\{x\}$, *modulus* $|x| = ((\text{Re}\{x\})^2 + (\text{Im}\{x\})^2)^{1/2}$, *phase* (argument) $\arg x = \arctan(\text{Im}\{x\}/\text{Re}\{x\})$, and *complex conjugate* $\bar{x} = \text{Re}\{x\} - i \text{Im}\{x\}$, where $i = \sqrt{-1}$. The entries of the amplitude vector $\mathbf{x} \in \mathbb{C}^n$ could be the amplitudes of distinct monochromatic sidebands (continuous waves) or different frequency components of multichromatic sidebands (pulses), with one or two polarization components [1], [14], [22], [23], [26], [27], [41].

Table 1.1 *Nomenclature. The equation number where a particular quantity first appears or is defined is listed, if appropriate.*

Notation	Description	Equation
\mathbf{a}	amplitude vector in two-mode case	(6.1)
A	skew-Hermitian matrix in \mathbf{x} system	(1.1a)
B	symmetric matrix in \mathbf{x} system	(1.1a)
C	arbitrary matrix, variously defined	
d	discriminant	
H	Hermitian matrix	(7.2)
i	indexing variable, variously defined	
i	as a subscript on \mathbf{a} , used to indicate idler	(6.1)
J	Hermitian block of L	(2.3)
j	indexing variable, variously defined	
K	symmetric block of L	(2.3)
L	matrix in \mathbf{y} system	(2.5b)
$M(z)$	transfer matrix in \mathbf{x} system	(1.1b)
$N(z)$	transfer matrix in \mathbf{x} system	(1.1b)
n	dimension of original system	
P	matrix in two-mode case	(7.1)
p	coupling coefficient in two-mode case	(7.6)
Q	orthogonal matrix	
R	symplectic matrix	(3.1)
S	matrix of eigenvectors of L	(2.8)
s	as a subscript on \mathbf{a} , used to indicate signal	(6.1)
T	as a superscript, used to indicate transpose	(2.1)
$U(z)$	unitary matrix	
\mathbf{u}	column of U	(4.4b)
$V(z)$	unitary matrix in SVD	(4.3)
\mathbf{v}	column of V	(4.4b)
\mathbf{w}	arbitrary vector, variously defined	
\mathbf{x}	n -dimensional vector of mode amplitudes	(1.1a)
\mathbf{y}	$2n$ -dimensional vector composed of \mathbf{x} and $\bar{\mathbf{x}}$	(2.5a)
\mathbf{z}	eigenvector, variously defined	
z	distance	(1.1a)
α	arbitrary constant, variously defined	
β	arbitrary constant, variously defined	
δ	self-coupling constant in two-mode case	(6.1)
Λ	diagonal matrix of eigenvalues of L	(2.8)
λ	eigenvalue, variously defined	
μ	eigenvalue of T	
$\Sigma(z)$	diagonal positive semidefinite matrix in SVD	(4.3)
σ	diagonal entry of Σ	(4.4a)
Ω	matrix used in definition of symplecticity	(3.1)
$*$	as a superscript, used to indicate conjugate transpose	(2.1)
$-$	as a superscript, used to indicate conjugate	(1.1a)

As (1.1a) is a linear equation, it may be formally solved to yield input-output relations (IORs, also called Bogolyubov transformations [2]) of the form

$$(1.1b) \quad \mathbf{x}(z) = M(z)\mathbf{x}(0) + N(z)\bar{\mathbf{x}}(0),$$

where $M, N \in \mathcal{C}^{n \times n}$ are transfer (Green) matrices found by solving the CMEs. If one can construct $M(z)$ and $N(z)$, then the problem is solved for all z , since the vectors $\mathbf{x}(0)$ and $\bar{\mathbf{x}}(0)$ are known.

For simple one- and two-mode interactions, it is easy to solve the CMEs and interpret the IORs. However, in some systems several modes interact simultaneously [30], or several two-mode interactions occur sequentially [32]. For such systems, the CMEs and IORs are more complicated. Hence two sets of questions arise that we shall address:

1. When can we solve the CMEs (1.1a) explicitly? What types of solutions are most desirable physically, and how can we achieve them?
2. As will be shown in the next section, the solution of the CMEs will be expressed in different terms than in the IORs. Under what general conditions can we relate these two expressions for the solution?

Answering these questions leads us to study the properties of the transfer matrices and other matrices defined below.

This module explores the relationship between the linear approximate form of the CMEs and the corresponding IORs. The spectral decomposition of the linear operator in the CMEs provides the solution in terms of the eigenvalues and eigenvectors of the underlying matrix; the form of those eigenvalues will provide insight into the physical structure. We will then relate the spectral decomposition of the CMEs to the SVD of the IORs, which decomposes the (z -dependent) matrices M and N into their singular values.

Section 2 explores the constraints on the coefficient matrices in (1.1a) which arise from linearizing the full nonlinear system. This exploration leads to the definition of several new matrices which are frequently discussed in the fiber-optics literature. Specifically, the matrices A and B are recast in terms of the Hermitian matrix J and the complex-symmetric matrix K , which are then used to define the matrices L , M , and N (the transfer matrices M and N were mentioned above). Again, all these definitions follow the traditions of fiber-optics research.

Section 3 then discusses perhaps the most important issue connecting the linear algebra under discussion with the overarching physics: the eigenvalues of the matrix L . Signal amplification requires that the system with constant pump and zero signal be unstable, and the stability of the system is determined by these eigenvalues. In particular, the eigenvalues of L occur in quartets, and the physical implications of this result for wave propagation are noted. Sections 4 and 5 then discuss the two decompositions of interest here: the spectral (or eigenvalue) decomposition and the singular value (or Schmidt) decomposition. Section 4 reviews the basic mathematical results for each decomposition; section 5 explores the relationship between the spectral (eigen)values and the singular values for a given matrix. This relationship is important because researchers discussing photon generation generally focus on the SVD since matrices arising in their work are not necessarily diagonalizable. However, the spectral decomposition is often easier to obtain and to relate to the physical problem. Unfortunately, however, the relationship between the spectral values and the singular values is quite limited.

In sections 6 and 7, the discussion returns more directly to fiber optics, in particular the implications of the results for the eigenvalues of L on isotropic and nonisotropic wave propagation through the optical fiber.

2. Coupled-Mode Equations. The laws of quantum mechanics impose constraints on the coefficient matrices in (1.1a), namely, that [24]

$$(2.1) \quad A = -A^*, \quad B = B^T.$$

Here A^* is the *conjugate transpose* of A , defined as $A^* = \bar{A}^T$ [16, section 6.4]. Also

note that B is a *symmetric* matrix. (Customarily the word “symmetric” is used only for real matrices. Here we expand the usage to complex matrices.)

One way to solve the problem is to note that the complex conjugate of (1.1a) is given by

$$(2.2) \quad \frac{d\bar{\mathbf{x}}}{dz} = \bar{A}\bar{\mathbf{x}} + \bar{B}\mathbf{x}.$$

Combining (1.1a) and (2.2), we have

$$\frac{d}{dz} \begin{pmatrix} \mathbf{x} \\ \bar{\mathbf{x}} \end{pmatrix} = \begin{pmatrix} A\mathbf{x} + B\bar{\mathbf{x}} \\ \bar{A}\bar{\mathbf{x}} + \bar{B}\mathbf{x} \end{pmatrix} = \begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \bar{\mathbf{x}} \end{pmatrix}.$$

In order to make more direct comparisons with the literature for the fiber-optic system under consideration, we let

$$(2.3) \quad A = iJ, \quad J = J^*; \quad B = iK, \quad K = K^T,$$

where we have used (2.1). Any matrix such as J that is equal to its own conjugate transpose is called a *Hermitian matrix* [16, section 6.4]. Hermitian matrices are the complex generalizations of real symmetric matrices. This is because, for complex matrices, the conjugate transpose has similar properties to real symmetric matrices with respect to the spectral theorem and its use in proofs.

Equation (2.3) allows us to rewrite our system as

$$(2.4) \quad \frac{d}{dz} \begin{pmatrix} \mathbf{x} \\ \bar{\mathbf{x}} \end{pmatrix} = \begin{pmatrix} iJ & iK \\ -i\bar{K} & -i\bar{J} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \bar{\mathbf{x}} \end{pmatrix},$$

$$\frac{d\mathbf{y}}{dz} = iL\mathbf{y},$$

$$(2.5a) \quad \mathbf{y} = \begin{pmatrix} \mathbf{x} \\ \bar{\mathbf{x}} \end{pmatrix},$$

$$(2.5b) \quad L = \begin{pmatrix} J & K \\ -\bar{K} & -\bar{J} \end{pmatrix}.$$

Note that we have expanded our system from n -dimensional to $2n$ -dimensional. Therefore, we seem to have introduced some additional degrees of freedom into the problem, but in fact the initial condition

$$(2.6) \quad \mathbf{y}(0) = \begin{pmatrix} \mathbf{x}(0) \\ \bar{\mathbf{x}}(0) \end{pmatrix}$$

takes care of that. Requiring that the last n elements of $\mathbf{y}(0)$ be the complex conjugates of the first n elements of $\mathbf{y}(0)$ provides the n additional conditions we need to close our $2n$ -dimensional system. (A more detailed proof is left to the reader; see Exercise 2.2.)

As long as L is independent of z , the solution of (2.4) subject to (2.6) can be formally expressed as

$$(2.7) \quad \mathbf{y}(z) = e^{izL}\mathbf{y}(0).$$

If L is diagonalizable, then the *spectral decomposition* of e^{izL} is given by

$$(2.8) \quad e^{izL} = Se^{iz\Lambda}S^{-1},$$

where S is a matrix of eigenvectors of L and Λ is the diagonal matrix of corresponding eigenvalues. Note that in addition to representing the forward evolution operator for y , the matrix exponential can also be expressed via the Maclaurin series of the exponential function. In this way, one sees immediately that if Λ is a diagonal matrix with

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

then $e^{iz\Lambda}$ is also diagonal with

$$e^{iz\Lambda} = \begin{pmatrix} e^{i\lambda_1 z} & 0 & \cdots & 0 \\ 0 & e^{i\lambda_2 z} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{i\lambda_n z} \end{pmatrix}.$$

Therefore, facts about the eigenvalues of L will tell us about the stability of the system. In particular, let $\lambda = \alpha + \beta i$ in (2.7). Then

$$(2.9) \quad e^{i\lambda z} = e^{(-\beta + \alpha i)z}.$$

Hence if one of the eigenvalues λ (say, λ_1) is real ($\beta = 0$), then the component of the solution in the \mathbf{z}_1 -direction will simply oscillate with respect to z :

$$(2.10) \quad \mathbf{y}(z) = c_1 e^{i\alpha_1 z} \mathbf{z}_1 + \cdots.$$

In general, it follows from (2.9) that the component will either grow or decay depending on the sign of β . In particular, if $\beta < 0$, the system generically admits a solution that exponentially grows in z . This instability in the amplitude is what characterizes signal amplification, which is one of the desirable experimental properties we are seeking. Moreover, if $\alpha = 0$, then the growth occurs without oscillation—another desirable property. Therefore, a key area to address is as follows.

QUESTION 1. *When does the matrix L have eigenvalues with negative imaginary part? When does it have a purely imaginary eigenvalue with negative imaginary part?*

Note that as the waves propagate down the optical fiber, we allow z to become arbitrarily large. Hence it may seem strange that exponential growth in z is allowed. For short distances, the pump waves provide enough energy to the signal and idler (sideband) waves to allow them to grow exponentially. For longer distances (and hence larger amplitudes), this process reverses and the sideband waves lose energy back to the pumps [5], [25].

This behavior can be seen mathematically by noting that the CMEs (1.1a) are simply the small-amplitude linearization of a more complicated system of the form

$$(2.11) \quad \frac{d\mathbf{x}}{dz} = A\mathbf{x} + B\bar{\mathbf{x}} - O(|\mathbf{x}|^2\mathbf{x}).$$

Note that when \mathbf{x} gets large in (2.11), the nonlinear term (which represents the loss of energy back to the pumps) becomes significant and can balance the linear terms, thereby arresting the exponential growth (see Figure 2.1). Physical examples of the case $n = 1$ are discussed in [24].

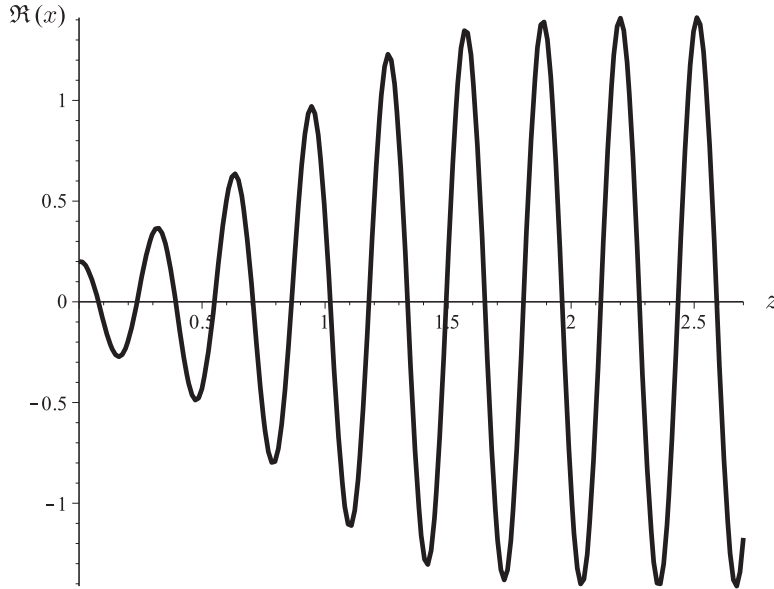


Fig. 2.1 Real part of the solution of $x' = (2 + 20i)x - |x|^2x$ vs. z . Note the amplitude at first grows (due to the coefficient 2), then asymptotes to a fixed value due to the nonlinear terms.

Though the spectral form of the solution as expressed in (2.7) is easily analyzable, recall that there is already another form of the solution given by the IORs (1.1b). Hence key insight into the problem can be obtained by relating the two representations. Therefore, another key question we will address is as follows.

QUESTION 2. *How are the matrices in the spectral representation (2.8) related to the transfer matrices in the IORs (1.1b)? How are the eigenvalues of L related to the singular values (defined below) of L ?*

We can partially answer Question 2 by combining (1.1b) and its complex conjugate. Rewriting in terms of \mathbf{x} , we obtain

$$(2.12) \quad \begin{pmatrix} \mathbf{x}(z) \\ \bar{\mathbf{x}}(z) \end{pmatrix} = \begin{pmatrix} M(z) & N(z) \\ \bar{N}(z) & \bar{M}(z) \end{pmatrix} \begin{pmatrix} \mathbf{x}(0) \\ \bar{\mathbf{x}}(0) \end{pmatrix},$$

$$e^{izL} = \begin{pmatrix} M(z) & N(z) \\ \bar{N}(z) & \bar{M}(z) \end{pmatrix}.$$

Noting that the matrix in (2.12) is always invertible with $e^{i(-z)L} = (e^{izL})^{-1}$, we have

$$(2.13) \quad \begin{pmatrix} M(z) & N(z) \\ \bar{N}(z) & \bar{M}(z) \end{pmatrix} \begin{pmatrix} M(-z) & N(-z) \\ \bar{N}(-z) & \bar{M}(-z) \end{pmatrix} = \begin{pmatrix} I & O \\ O & I \end{pmatrix},$$

which gives us relations between M and N of positive and negative argument. (I is the $n \times n$ identity matrix.) Moreover, from the laws of quantum mechanics we have that [7], [24]

$$(2.14a) \quad MM^* - NN^* = I,$$

$$(2.14b) \quad MN^T - NM^T = O.$$

Equations (2.14) are true whether or not L depends on z . However, for our special case where L is independent of z , (2.13) and (2.14) yield the relation in the following lemma.

LEMMA 2.1. *Let L be independent of z in (2.4). Then*

$$(2.15) \quad M(-z) = M^*(z), \quad \bar{N}(-z) = -N^*(z) \quad \implies \quad N(-z) = -N^T(z).$$

Proof. Exercise 2.3.

Hence the spectral representation provides key restrictions on the forms of the transfer matrices. In particular, certain forms which violate (2.15) may be ruled out.

We conclude this section by examining another facet of Question 2. We specialize to the case where B equals the zero matrix O (the $n \times n$ matrix whose elements are all zero). In this case, (1.1a) reduces to a standard system of ODEs, $N = O$, and from (1.1) we have that $M = e^{zA}$. Moreover, from (2.14) we have that $MM^* = I$. Such a matrix is called *unitary*, which is the generalization of an orthogonal matrix to the complex case. The fact that M is unitary may also be shown by using the interpretation of M as a matrix exponential (see Exercise 2.4).

Since M is a unitary matrix, all its eigenvalues have modulus 1 (that is, they lie on the unit circle in the complex plane). Moreover, the columns of M are orthogonal [16, section 6.4].

EXERCISE 2.1. *Show that the trace of L is 0. (Hint: What property must the diagonal entries of a Hermitian matrix have?)*

EXERCISE 2.2. *Show that with L as defined in (2.5b), the solution of (2.4) will always be of the form (2.5a) as long as the initial condition is of the form (2.6).*

EXERCISE 2.3. *Prove Lemma 2.1. First use (2.12) to show that either M or N must be invertible, then use this fact to show that assuming the lemma not to be true leads to a contradiction.*

EXERCISE 2.4. *In the case that $B = O$, we know that $M = e^{zA}$ from our previous remarks. Use (2.1) to show that M is unitary in this case.*

3. Eigenvalues of L . In this section we derive the following result, which provides some insight into the answer to Question 1.

THEOREM 3.1. *The eigenvalues of L come in (perhaps degenerate) quartets: $\{\lambda, -\bar{\lambda}, \bar{\lambda}, -\lambda\}$.*

Theorem 3.1 then guarantees that if L has a complex eigenvalue, then it must have one with negative imaginary part. As discussed above, this corresponds to the case where the pump waves provide energy to the sideband waves in such a way as to force purely exponential growth, guaranteeing signal amplification. The case of a real eigenvalue produces degeneracy (since $\lambda = \bar{\lambda}$, the set is just two real eigenvalues), as does a purely imaginary eigenvalue (since $\lambda = -\bar{\lambda}$, the set is just two imaginary eigenvalues).

We present two proofs of Theorem 3.1. Each proof utilizes different techniques and illustrates various aspects of the problem.

3.1. Proof by Symplectic Structure. In this section we prove Theorem 3.1 using only theoretical properties of L and e^{izL} . First, we deduce that e^{izL} is symplectic. A matrix $R \in \mathcal{C}^{2n \times 2n}$ is called *symplectic* if [21]

$$(3.1) \quad R\Omega R^T = \Omega, \quad \Omega = \begin{pmatrix} O & I \\ -I & O \end{pmatrix}, \quad I \in \mathcal{C}^{n \times n}.$$

Symplectic transformations arise naturally in both classical [9, section 9.3] and quantum [18] mechanics.

LEMMA 3.2. e^{izL} is symplectic.

Proof. Exercise 3.2.

COROLLARY 3.2.1. If λ is an eigenvalue for L , so is $-\lambda$.

Proof. Let R be a symplectic matrix, and let μ be an eigenvalue of R^T (and hence R) corresponding to \mathbf{z} . Then

$$\Omega \mathbf{z} = R \Omega R^T \mathbf{z} = \mu R(\Omega \mathbf{z}).$$

Hence μ^{-1} is an eigenvalue of R . In our case,

$$R = e^{izL},$$

so $\mu = e^{i\lambda z}$ for some eigenvalue λ of L . Hence $\mu^{-1} = e^{i(-\lambda)z}$ and corresponds to a second eigenvalue $-\lambda$ of L . \square

The second part of the proof comes from properties of the determinant of a partitioned matrix. Let

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}.$$

Then the determinant remains the same if the sign of either the diagonal or off-diagonal blocks are changed.

LEMMA 3.3. $\det C = \det C' = \det(-C')$, where

$$C' = \begin{pmatrix} I & O \\ O & -I \end{pmatrix} \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} I & O \\ O & -I \end{pmatrix} = \begin{pmatrix} C_{11} & -C_{12} \\ -C_{21} & C_{22} \end{pmatrix}.$$

Proof. Exercise 3.3.

COROLLARY 3.3.1. If λ is an eigenvalue for L , so is $\bar{\lambda}$.

Proof. We wish to consider the following two matrices:

$$L_\lambda = \begin{pmatrix} J - \lambda I & K \\ -\bar{K} & -\bar{J} - \lambda I \end{pmatrix}, \quad L'_\lambda = \begin{pmatrix} J - \lambda I & -K \\ \bar{K} & -\bar{J} - \lambda I \end{pmatrix}.$$

By Lemma 3.3, $\det L_\lambda = \det L'_\lambda$. Therefore, if λ is an eigenvalue for L (and hence $\det L_\lambda = 0$), then $\det L'_\lambda = 0$, so λ is an eigenvalue for

$$\begin{pmatrix} J & -K \\ \bar{K} & -\bar{J} \end{pmatrix} = \begin{pmatrix} J & -K^T \\ K^* & -\bar{J} \end{pmatrix} = L^*.$$

However, the eigenvalues of L^* are the complex conjugates of the eigenvalues of L , so we have $\lambda = \bar{\lambda}$. \square

3.2. Proof by Eigenvector Construction. In the previous section we proved Theorem 3.1 using theoretical properties of matrices. However, from the engineering perspective it would be useful to have some knowledge of the eigenvectors of L , since they characterize the modes that will experience signal amplification. Therefore, in this section we provide a more constructive proof using facts about the eigenvectors of L . We begin with the following lemma.

LEMMA 3.4. Let

$$(3.2) \quad L\mathbf{z} = \lambda\mathbf{z}, \quad \mathbf{z} = \begin{pmatrix} \mathbf{y}_a \\ \mathbf{y}_b \end{pmatrix}.$$

Then

$$(3.3a) \quad L\mathbf{z}_1 = -\bar{\lambda}\mathbf{z}_1, \quad \mathbf{z}_1 = \begin{pmatrix} \bar{\mathbf{y}}_b \\ \bar{\mathbf{y}}_a \end{pmatrix},$$

$$(3.3b) \quad L^* \begin{pmatrix} \mathbf{y}_a \\ -\mathbf{y}_b \end{pmatrix} = \lambda \begin{pmatrix} \mathbf{y}_a \\ -\mathbf{y}_b \end{pmatrix}.$$

Proof. We note from the hypothesis that

$$(3.4) \quad L \begin{pmatrix} \mathbf{y}_a \\ \mathbf{y}_b \end{pmatrix} = \lambda \begin{pmatrix} \mathbf{y}_a \\ \mathbf{y}_b \end{pmatrix} \quad \implies \quad \begin{aligned} J\mathbf{y}_a + K\mathbf{y}_b &= \lambda\mathbf{y}_a, \\ -\bar{K}\mathbf{y}_a - \bar{J}\mathbf{y}_b &= \lambda\mathbf{y}_b. \end{aligned}$$

To establish (3.3b), we note that

$$L^* \begin{pmatrix} \mathbf{y}_a \\ -\mathbf{y}_b \end{pmatrix} = \begin{pmatrix} J^* & -K^T \\ K^* & -J^T \end{pmatrix} \begin{pmatrix} \mathbf{y}_a \\ -\mathbf{y}_b \end{pmatrix} = \begin{pmatrix} J\mathbf{y}_a + K\mathbf{y}_b \\ \bar{K}\mathbf{y}_a + \bar{J}\mathbf{y}_b \end{pmatrix} = \begin{pmatrix} \lambda\mathbf{y}_a \\ \lambda(-\mathbf{y}_b) \end{pmatrix} = \lambda \begin{pmatrix} \mathbf{y}_a \\ -\mathbf{y}_b \end{pmatrix},$$

where we have used (2.3) and (3.4). The proof of (3.3a) is left as Exercise 3.4 for the reader. \square

Note that the lemma says nothing about how to determine \mathbf{y}_a and \mathbf{y}_b . However, if we did obtain them, the lemma indicates how the resulting eigenvector is related to other eigenvectors for L and L^* .

Proof of Theorem 3.1. The first two elements of the quartet follow directly from Lemma 3.4. Moreover, we know that if λ is an eigenvalue for L^* , then $\bar{\lambda}$ is an eigenvalue for L . The last member of the quartet results from applying the first part of Lemma 3.4 to $\bar{\lambda}$. \square

As discussed above, growth occurs without oscillation if L has a purely imaginary eigenvalue. This causes purely exponential growth, which represents the strongest possible transfer of energy from the pump field to the sidebands for eigenvalues with fixed modulus $|\lambda|$. This physically desirable case is related to the degenerate case of Theorem 3.1, as discussed in section 3.1. The following two corollaries of Theorem 3.1 provide examples of occasions where degeneracies occur.

COROLLARY 3.1.1. *If n is odd, there exist at least two purely real or two purely imaginary eigenvalues in \pm pairs.*

Proof. Exercise 3.5.

COROLLARY 3.1.2. *Let $\mathbf{y}_a = \alpha\bar{\mathbf{y}}_b$, where α is a complex constant with $|\alpha| = 1$. Then L has an imaginary eigenvalue.*

Proof. In this case, (3.2) and (3.3a) are equivalent (see Exercise 3.6). Hence $\lambda = -\bar{\lambda}$ and λ is imaginary. \square

Note that when we substitute our quartet elements into the spectral decomposition, we obtain

$$e^{i(-\bar{\lambda})z} = e^{\overline{(i\lambda)}} = \overline{(e^{i\lambda z})}.$$

Hence the two eigenvalues related directly through Lemma 3.4 correspond to complex conjugate pairs, as one would expect if L were real. However, conjugate pairs do not generally occur for arbitrary complex matrices; hence L is of special character.

EXERCISE 3.1. *Verify that the result of Theorem 3.1 is consistent with Exercise 2.1.*

EXERCISE 3.2. *Prove Lemma 3.2 by direct substitution using (2.12) and (2.14).*

EXERCISE 3.3. *Prove Lemma 3.3.*

EXERCISE 3.4. *Prove the validity of (3.3a).*

EXERCISE 3.5. *Prove Corollary 3.1.1.*

EXERCISE 3.6. *Show that under the assumptions in Corollary 3.1.2, \mathbf{z} and \mathbf{z}_1 as defined in Lemma 3.4 are proportional to one another.*

4. Decompositions. In order to better address Question 2 (which is done more fully in section 5), in this section we examine each of the two types of decompositions in detail.

4.1. The Spectral Decomposition. Recall that, in general, writing $C = S\Lambda S^{-1}$ has an attractive physical interpretation when computing the matrix-vector product $C\mathbf{x} = S\Lambda S^{-1}\mathbf{x}$. In particular, $S^{-1}\mathbf{x}$ transforms the vector \mathbf{x} to the eigenvector basis. In our case, the eigenvectors consist of modes of the solution in that the action of C , given by $\Lambda S^{-1}\mathbf{x}$, consists of dilations and phase rotations along the eigenvectors. The dilation factors (given by the eigenvalues) illustrate how effectively the energy is transmitted from the pump field to the sidebands. Finally, the product $\Lambda S^{-1}\mathbf{x}$ is multiplied by S to transform it back into the original (physical) basis. For more details, see [15, section 5.4].

It is well known that a real symmetric matrix C has a complete set of orthogonal eigenvectors, and hence by the spectral decomposition theorem it can be orthogonally diagonalized [39, section 5.3]:

$$C = C^T \quad \implies \quad C = Q\Lambda Q^T, \quad QQ^T = I.$$

This is a very convenient property to have, because the orthogonality of the eigenvectors simplifies various calculations (in our case, related to the strength of various modes of the signal).

The complex analogue of this is a matrix which is *unitarily diagonalizable*, i.e., its matrix of eigenvectors forms a unitary matrix:

$$C = U\Lambda U^*, \quad UU^* = I.$$

The most general class of unitarily diagonalizable matrices is that of the *normal matrices*, which are defined as those which commute with their complex conjugate transpose:

$$(4.1) \quad C^*C = CC^*.$$

Note that Hermitian matrices are automatically normal (see Exercise 4.1).

Since it is desirable for the matrix L under analysis to be unitarily diagonalizable, we derive the conditions under which this is true, using the matrices A and B from (1.1a).

LEMMA 4.1. *L is normal (and hence can be unitarily diagonalized) if and only if AB is antisymmetric.*

Proof. Using (2.3), we have that

$$(4.2) \quad \begin{aligned} B^T A^T &= B(-A^*)^T = -B\bar{A} = -(iK)(\overline{iJ}) = -K\bar{J}, \\ AB &= (iJ)(iK) = -JK. \end{aligned}$$

If AB is antisymmetric, then $K\bar{J} = -JK$. The rest of the proof is left as Exercise 4.2. \square

Clearly the requirement in Lemma 4.1 is very restrictive, and hence in general L is not normal. Even in that case, L may still be diagonalizable, but not by a unitary matrix.

4.2. The Singular Value Decomposition. Since the diagonalizability of L cannot be guaranteed, physicists often like to work with the transfer matrices on the right-hand side of (1.1b), expressing them in a singular value (or Schmidt) decomposition, rather than the spectral decomposition. (Every matrix has an SVD.)

The SVD for a matrix M is given by

$$(4.3) \quad M = U_M \Sigma_M V_M^*,$$

where U_M and V_M are unitary matrices, and Σ_M is a diagonal matrix. The diagonal entries σ_j of Σ_M (called the *singular values* of M) are the nonnegative square roots of the eigenvalues of M^*M . One can show that M^*M has all nonnegative eigenvalues, and as such it is an example of a *positive semidefinite* matrix [39, section 6.5].

The SVD has a similar interpretation to the spectral decomposition when computing the matrix-vector product $M\mathbf{x} = U_M \Sigma_M V_M^* \mathbf{x}$. In particular, $V_M^* \mathbf{x}$ transforms the vector \mathbf{x} to a basis where the multiplication by M simply consists of multiplication by the singular values, which are never negative. Then the resulting product $\Sigma_M V_M^* \mathbf{x}$ is multiplied by U_M to transform it into yet another basis. Hence the input and output bases under an SVD are different, which is the additional mathematical complication caused by relaxing the diagonalizability requirement. However, the change of basis may have physical relevance in the optical context.

Though the SVD has many useful properties, only a few are relevant to the discussion here (for more details, see [16, section 6.5] or [40]). In the SVD, the matrix Σ_M is unique because, by convention,

$$(4.4a) \quad \sigma_1 \geq \sigma_2 \geq \cdots.$$

Moreover, the columns of U_M and V_M are related as follows:

$$(4.4b) \quad A\mathbf{v}_j = \sigma_j \mathbf{u}_j, \quad A^* \mathbf{u}_j = \sigma_j \mathbf{v}_j.$$

Hence while the matrices U_M and V_M are not unique, they are severely restricted. In particular, note that since U_M and V_M are unitary, each column must be a unit vector, so if we wish to change one column \mathbf{u}_j to a new column \mathbf{u}'_j , we must also change \mathbf{v}_j according to the relation

$$(4.5) \quad \mathbf{u}'_j = e^{i\alpha} \mathbf{u}_j \quad \implies \quad \mathbf{v}'_j = e^{i\alpha} \mathbf{v}_j,$$

where $\alpha \in \mathcal{R}$.

The quantum mechanical relations (2.14) provide a simplifying relationship between the SVDs of M and N . First, we have that

$$(4.6) \quad N = U_M \Sigma_N V_M^T.$$

Though the proof of this relationship is somewhat involved [4], [24], it can be shown to be consistent by substitution (see Exercise 4.3).

Note that the spectral decomposition (2.8) has the convenient attribute that all the z -dependence of the transfer matrix is confined to the diagonal matrix $e^{iz\Lambda}$. In contrast, the SVDs in (4.3) and (4.6) generally have z -dependence in all three matrices [24]. Hence the input and output bases will change with distance, reflecting the changing properties of the optical fiber.

We may also obtain a relationship between the singular values of M and N .

LEMMA 4.2. *Let M and N be transfer matrices with the SVDs given in (4.3) and (4.6). Then*

$$(4.7) \quad \Sigma_M^2 - \Sigma_N^2 = I.$$

Proof. The result follows directly from substituting (4.3) and (4.6) into (2.14a):

$$\begin{aligned} (U_M \Sigma_M V_M^*)(U_M \Sigma_M V_M^*)^* - (U_M \Sigma_N V_M^T)(U_M \Sigma_N V_M^T)^* &= I, \\ U_M \Sigma_M^2 U_M^* - U_M \Sigma_N^2 U_M^* &= U_M U_M^*, \\ U_M (\Sigma_M^2 - \Sigma_N^2 - I) U_M^* &= O. \quad \square \end{aligned}$$

With this result, we can now begin to relate the two decompositions. In particular, we may write the SVD of the full $2n$ -dimensional system (which generates the spectral decomposition) in terms of the SVDs of the transfer matrices:

THEOREM 4.3. *e^{izL} can be decomposed as $e^{izL} = U \Sigma V^*$, where*

$$(4.8a) \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} U_M & U_M \\ \bar{U}_M & -\bar{U}_M \end{pmatrix}, \quad V = \frac{1}{\sqrt{2}} \begin{pmatrix} V_M & V_M \\ \bar{V}_M & -\bar{V}_M \end{pmatrix},$$

$$(4.8b) \quad \Sigma = \begin{pmatrix} \Sigma_M + \Sigma_N & O \\ O & \Sigma_M - \Sigma_N \end{pmatrix}.$$

Proof. Exercise 4.4. \square

As written, this decomposition is nearly an SVD. We may verify that U is unitary:

$$U^* U = \frac{1}{2} \begin{pmatrix} U_M^* & -U_M^T \\ U_M^* & U_M^T \end{pmatrix} \begin{pmatrix} U_M & U_M \\ -\bar{U}_M & \bar{U}_M \end{pmatrix} = \frac{1}{2} \begin{pmatrix} I + \overline{(U_M^* U_M)} & I - \overline{(U_M^* U_M)} \\ I - \overline{(U_M^* U_M)} & I + \overline{(U_M^* U_M)} \end{pmatrix} = I,$$

where we have used the fact that U_M and V_M are unitary. (This explains the introduction of the $\sqrt{2}$ terms in (4.8a).) A similar computation holds for V . From (4.7) we note that Σ may also be written as

$$(4.9) \quad \Sigma = \begin{pmatrix} \Sigma_M + \Sigma_N & O \\ O & (\Sigma_M + \Sigma_N)^{-1} \end{pmatrix},$$

which reinforces the fact that all the diagonal entries in Σ are positive.

So why is this decomposition not an SVD? Under the formal definition of an SVD, the singular values must be ordered as in (4.4a). As written in (4.9), the smallest entry of Σ is $\sigma_{n+1} = \sigma_1^{-1}$, since σ_1 is the sum of the largest entries of Σ_M and Σ_N by (4.4a). However, this ordering violates (4.4a). For our purposes, the ordering is unimportant, so we may analyze this decomposition as if it were a true SVD. Moreover, it can be shown (see Exercise 4.5) that the entries of Σ , U , and V may be permuted so that we obtain an authentic SVD.

We conclude with a final result about the SVD that we shall use in later sections.

LEMMA 4.4. *K has an SVD of the form $K = U_K \Sigma_K U_K^T$, where U_K is unitary.*

Proof. From (4.4b) we have that

$$(4.10) \quad K \mathbf{v}_j = \sigma_j \mathbf{u}_j, \quad K^* \mathbf{u}_j = \sigma_j \mathbf{v}_j,$$

where \mathbf{u}_j and \mathbf{v}_j are the columns of U_K and V_K , the unitary matrices in the SVD of K . Taking the complex conjugate of the last equation, we obtain

$$(4.11) \quad \begin{aligned} K^T \bar{\mathbf{u}}_j &= \bar{\sigma}_j \bar{\mathbf{v}}_j, \\ K \bar{\mathbf{u}}_j &= \sigma_j \bar{\mathbf{v}}_j, \end{aligned}$$

where we have used the facts that K is symmetric and σ_j is real. Therefore, by our discussion after (4.4), we see that $\bar{\mathbf{u}}_j = e^{i\beta} \mathbf{v}_j$. Then, making the substitution in (4.5), we have

$$\bar{\mathbf{u}}'_j = \overline{e^{i\alpha} \mathbf{u}_j} = e^{-i\alpha} e^{i\beta} \mathbf{v}_j = \frac{e^{i(\beta-\alpha)}}{e^{i\alpha}} \mathbf{v}'_j.$$

Hence by choosing $\alpha = \beta/2$, we have that $\bar{\mathbf{u}}'_j = \mathbf{v}'_j$. This makes (4.11) consistent with the first equation in (4.10). Hence (now dropping the primes), $V_K = \bar{U}_K$ and $K = U_K \Sigma_K (\bar{U}_K)^* = U_K \Sigma_K U_K^T$. \square

Since the SVD is not unique, the choice in Lemma 4.4 is only one of many possible, but it is the most convenient form for our work in section 6, where we will answer Question 1 regarding the eigenvalues of the specific case of two-mode isotropic propagation.

EXERCISE 4.1. *Prove that Hermitian matrices are normal.*

EXERCISE 4.2. *Complete the proof of Lemma 4.1 by computing both sides of (4.1) for L .*

EXERCISE 4.3. *Verify that (4.6) satisfies (2.14b). (Hint: Diagonal matrices always commute.)*

EXERCISE 4.4. *Prove Theorem 4.3 by direct substitution.*

EXERCISE 4.5. *Denote the j th column of U (as defined in (4.8a)) by \mathbf{u}_j . Show that if we define*

$$U' = (\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n, \mathbf{u}_{2n}, \mathbf{u}_{2n-1}, \dots, \mathbf{u}_{n+1})$$

and permute Σ and V (as defined in (4.8a)) similarly, then

$$e^{izL} = U' \Sigma' (V')^*$$

is an SVD with proper ordering.

5. Relating the Decompositions. Now we turn our attention to answering Question 2. Unfortunately, we must answer it mostly in the negative; that is, beyond Theorem 4.3, explicit relationships occur in very limited special cases. There is, however, one general relationship that does hold.

PROPOSITION 5.1. *Let C be a general diagonalizable matrix with SVD $C = U \Sigma V^*$ and spectral decomposition $C = S \Lambda S^{-1}$. Then*

$$|\det C| = |\det \Sigma| = |\det \Lambda|.$$

Proof. Exercise 5.1.

Recall that for our purposes, $C = e^{izL}$, which is related to the transfer matrices M and N through (2.12).

Beyond this general result, the key issue is symmetry: If $C = C^*$, then its eigenvectors are orthogonal, implying that $S^{-1} = S^*$. So in this case, S is unitary and $C = S \Lambda S^*$. However, the SVD and the spectral decomposition are *not* the same in this case, as described in Proposition 5.2 below. Note that a *skew-Hermitian* matrix C has $C = -C^*$ and is the complex generalization of an antisymmetric matrix. Hence its eigenvalues are pure imaginary [16, section 6.4]. Recall that pure imaginary eigenvalues correspond to exponential growth and maximal transfer of energy from the pump field to the sidebands.

PROPOSITION 5.2. *If C is Hermitian or skew-Hermitian, then $\sigma_j = |\lambda_j|$ for some appropriate ordering of eigenvalues. If C is also positive semidefinite, then $\sigma_j = \lambda_j$.*

Proof. If C is Hermitian, the entries of Σ are the nonnegative square roots of the eigenvalues of $CC^* = C^2$. So $\sigma = \sqrt{\lambda_{C^2}} = \sqrt{\lambda_C^2} = |\lambda|$. If C is also positive semidefinite, $\lambda \geq 0$, so $|\lambda| = \lambda$. Similarly, if C is skew-Hermitian, then $C^*C = -C^2$, so $\sigma = \sqrt{\lambda_{-C^2}} = \sqrt{-\lambda_C^2} = |\lambda|$. \square

But if C is not real symmetric or Hermitian, no general results beyond Proposition 5.1 seem possible. For instance, consider the case where

$$U_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad V_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Then

$$C_1 = U_1 \Sigma V_1^* = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix},$$

but also $C_1 = S_1 \Lambda_1 S_1^{-1}$, where

$$S_1 = \begin{pmatrix} \sqrt{2}i & \sqrt{2}i \\ -1 & 1 \end{pmatrix}, \quad S_1^{-1} = \frac{-i}{2\sqrt{2}} \begin{pmatrix} 1 & -\sqrt{2}i \\ 1 & \sqrt{2}i \end{pmatrix}, \quad \Lambda_1 = \begin{pmatrix} -\sqrt{2}i & 0 \\ 0 & \sqrt{2}i \end{pmatrix}.$$

For this C_1 , the elements of Σ and Λ_1 are unique (up to order), so clearly the elements of these two matrices do not have the same modulus, are not all real, and seem related only in that they have the same product (the same determinant).

In the above example, C was not symmetric. However, the same problem in relating the eigenvalues and singular values occurs if C is complex and symmetric, but not Hermitian (see Exercise 5.2).

EXERCISE 5.1. *Prove Proposition 5.1.*

EXERCISE 5.2. *Construct a complex symmetric (but non-Hermitian) matrix C . Compute its spectral and singular value decompositions, and show that the related matrices do not share any particular characteristics those in Proposition 5.1.*

6. Two-Mode Case, Isotropic Propagation. In order to answer Question 1 more explicitly in the case of an actual experimental system, we specialize to the case of two-mode interaction. This substantially simplifies the problem and allows explicit computations.

First, we consider the case of linear, uncoupled wave propagation (e.g., two polarization components in a nonbirefringent fiber, or many frequency components in a nondispersive fiber) [14], [23], [26], [27]. In that case, all the components of the signal amplitude \mathbf{a}_s and idler amplitude \mathbf{a}_i experience the same type of propagation. Mathematically, this may be expressed as

$$(6.1) \quad \frac{d}{dz} \begin{pmatrix} \mathbf{a}_s \\ \mathbf{a}_i \end{pmatrix} = iL \begin{pmatrix} \mathbf{a}_s \\ \mathbf{a}_i \end{pmatrix}, \quad L = \begin{pmatrix} \delta I & K \\ -\bar{K} & -\delta I \end{pmatrix}.$$

Here the δI term models the isotropic propagation, and K models the interaction among the pump field and the sidebands (\mathbf{a}_s and \mathbf{a}_i). Note that (6.1) is in exactly the same form as (2.4) and (2.5b) with $J = \delta I$, and so the results from the previous sections hold.

In order to determine the strength of signal amplification in this system, we must examine the eigenvalues. Recalling that A and B are the coefficient matrices from the CMEs (1.1a), we begin by considering a much more general case.

LEMMA 6.1. *Let AB be symmetric, and let \mathbf{u}_K be a column of U_K (the matrix in the SVD of K). Then the following hold:*

1. \mathbf{u}_K is an eigenvector for J .
2. There exist constants β, λ_L such that

$$(6.2) \quad L \begin{pmatrix} \beta \mathbf{u}_K \\ \bar{\mathbf{u}}_K \end{pmatrix} = \lambda_L \begin{pmatrix} \beta \mathbf{u}_K \\ \bar{\mathbf{u}}_K \end{pmatrix}.$$

Proof. First, define

$$(6.3) \quad \mathbf{w}_K = J\mathbf{u}_K.$$

Recall from Lemma 4.4 that $K\bar{\mathbf{u}}_K = \sigma_K \mathbf{u}_K$. Using this fact, we compute the following quantities:

$$\begin{aligned} (JK)\bar{\mathbf{u}}_K &= J(\sigma_K \mathbf{u}_K) = \sigma_K J\mathbf{u}_K = \sigma_K \mathbf{w}_K, \\ (K\bar{J})\bar{\mathbf{u}}_K &= K(\bar{J}\bar{\mathbf{u}}_K) = K\bar{\mathbf{w}}_K. \end{aligned}$$

By analogy with the proof of Lemma 4.1, these two quantities are equal if AB is symmetric. Hence $\bar{\mathbf{w}}_K$ must be proportional to $\bar{\mathbf{u}}_K$, as long as all the singular values are distinct. (The statement is also true for the indistinct case, but the proof is beyond the scope of this article.) Defining the constant of proportionality as $\bar{\lambda}_J$, we have

$$\bar{\mathbf{w}}_K = \bar{\lambda}_J \bar{\mathbf{u}}_K \quad \implies \quad J\mathbf{u}_K = \lambda_J \mathbf{u}_K,$$

where we have used (6.3). Hence item 1 is proved. Performing the multiplication in (6.2), we obtain

$$L \begin{pmatrix} \beta \mathbf{u}_K \\ \bar{\mathbf{u}}_K \end{pmatrix} = \begin{pmatrix} J & K \\ -\bar{K} & -\bar{J} \end{pmatrix} \begin{pmatrix} \beta \mathbf{u}_K \\ \bar{\mathbf{u}}_K \end{pmatrix} = \begin{pmatrix} (\beta \lambda_J + \sigma_K) \mathbf{u}_K \\ -\beta \bar{K} \mathbf{u}_K - \lambda_J \bar{\mathbf{u}}_K \end{pmatrix},$$

but $\bar{K} \mathbf{u}_K = \overline{(K \bar{\mathbf{u}}_K)} = \overline{(\sigma_K \mathbf{u}_K)} = \sigma_K \bar{\mathbf{u}}_K$, since σ_K is real. Continuing to simplify, we have

$$(6.4) \quad L \begin{pmatrix} \beta \mathbf{u}_K \\ \bar{\mathbf{u}}_K \end{pmatrix} = \begin{pmatrix} (\lambda_J + \sigma_K/\beta)(\beta \mathbf{u}_K) \\ -(\beta \sigma_K + \lambda_J) \bar{\mathbf{u}}_K \end{pmatrix}.$$

Therefore, our theorem is true if and only if

$$\begin{aligned} \lambda_L &= \lambda_J + \frac{\sigma_K}{\beta} = -(\beta \sigma_K + \lambda_J), \\ \sigma_K \beta^2 + 2\lambda_J \beta + \sigma_K &= 0, \\ (6.5a) \quad \beta &= \frac{-2\lambda_J \pm \sqrt{4\lambda_J^2 - 4\sigma_K^2}}{2\sigma_K} = \frac{-\lambda_J \pm \sqrt{\lambda_J^2 - \sigma_K^2}}{\sigma_K}, \end{aligned}$$

$$(6.5b) \quad \lambda_L = \mp \sqrt{\lambda_J^2 - \sigma_K^2}. \quad \square$$

In this case, more general than (6.1), the eigenvalues of L will be either purely real or purely imaginary. Equation (6.5b) exhibits a competition between the action of J , represented by the eigenvalues λ_J , and that of K , represented by its singular values σ_K . Whereas the isotropic propagation induced by J encourages the signal to rotate its polarization with no transfer of energy, the impact of the pump fields is contained in K . If sufficiently strong ($\sigma_K > \lambda_J$), this influence forces the eigenvalues onto the imaginary axis, implying instability and therefore signal growth.

Remarks.

1. Note that in contrast to Lemma 4.1, which is true if AB is antisymmetric, Lemma 6.1 is true if AB is symmetric.
2. Recall that U_K is $n \times n$. Hence the fact that there are two choices for β and λ_L for each of the n \mathbf{u}_K vectors produces a full set of $2n$ eigenvalues and eigenvectors for L .
3. Since U_K is unitary, the vectors \mathbf{u}_K are orthogonal. Hence, denoting the i th column of U_K by $\mathbf{u}_{K,i}$, we have the following:

$$\begin{pmatrix} \beta_i \mathbf{u}_{K,i} \\ \bar{\mathbf{u}}_{K,i} \end{pmatrix}^* \begin{pmatrix} \beta_j \mathbf{u}_{K,j} \\ \bar{\mathbf{u}}_{K,j} \end{pmatrix} = 0, \quad i \neq j.$$

Thus the eigenvector of L constructed from $\mathbf{u}_{K,i}$ will be orthogonal to the $2(n-1)$ eigenvectors of L constructed from $\mathbf{u}_{K,j}$. It will not be orthogonal only to the second eigenvector of L constructed from the same \mathbf{u}_K . Hence calculating the underlying modes is simpler, since the matrix is “nearly unitary.”

Now we consider the specific system (6.1). First, we note from (6.2) that both signal and idler have the same basis vectors \mathbf{u} . We can calculate the eigenvalues directly.

COROLLARY 6.1.1. *The eigenvalues and eigenvectors of the system (6.1) are given by (6.5) with*

$$\beta = \frac{-\delta \pm \sqrt{\delta^2 - \sigma_K^2}}{\sigma_K}, \quad \lambda_L = \mp \sqrt{\delta^2 - \sigma_K^2}.$$

Proof. As discussed previously, AB being symmetric is equivalent to $JK = K\bar{J}$. With $J = \delta I$, this is clearly the case. The remainder of the proof is left as Exercise 6.1. \square

The case where K is real is quite similar, with eigenvalues replacing singular values.

LEMMA 6.2. *Let K be real in the definition of L in (2.5b), AB be symmetric, and $K\mathbf{z}_K = \lambda_K\mathbf{z}_K$. Then the following hold.*

1. *Both λ_K and \mathbf{z}_K are real, and the eigenvectors are orthogonal.*
2. *There exist constants β , λ_L such that*

$$(6.6) \quad L \begin{pmatrix} \beta \mathbf{z}_K \\ \mathbf{z}_K \end{pmatrix} = \lambda_L \begin{pmatrix} \beta \mathbf{z}_K \\ \mathbf{z}_K \end{pmatrix}.$$

Proof. Since K is now real and symmetric, it has real eigenvalues and a full set of real orthonormal eigenvectors, so item 1 is proved. Since K is real, we have from the proof of Lemma 4.4 that $JK = KJ$. Hence they have the same eigenvectors [39, p. 321, Ex. 44], so $J\mathbf{z}_K = \lambda_J\mathbf{z}_K$ for λ_J not necessarily equal to λ_K . The rest of the proof is left as Exercise 6.2; the values in question are

$$\beta = \frac{-\lambda_J \pm \sqrt{\lambda_J^2 - \lambda_K^2}}{\lambda_K}, \quad \lambda_L = \mp \sqrt{\lambda_J^2 - \lambda_K^2}. \quad \square$$

COROLLARY 6.2.1. *If K is real, the eigenvalues and eigenvectors of the system (6.1) are given by (6.6) with*

$$\beta = \frac{-\delta \pm \sqrt{\delta^2 - \lambda_K^2}}{\lambda_K}, \quad \lambda_L = \mp \sqrt{\delta^2 - \lambda_K^2}.$$

Proof. The result follows trivially from Lemma 6.2 and Corollary 6.1.1. \square

We use the same sort of technique to derive the factorization in (4.8) in a different way.

LEMMA 6.3. *Let M and N be transfer matrices with $M\mathbf{v}_M = \sigma_M\mathbf{u}_M$ for some $\sigma_M \geq 0$. (So σ_M is the singular value and \mathbf{u}_M and \mathbf{v}_M are columns of the matrices U_M, V_M in (4.3).) Then*

$$(6.7) \quad e^{izL} \begin{pmatrix} \mathbf{v}_M \\ \bar{\mathbf{v}}_M \end{pmatrix} = (\sigma_M + \sigma_N) \begin{pmatrix} \mathbf{u}_M \\ \bar{\mathbf{u}}_M \end{pmatrix}, \quad e^{izL} \begin{pmatrix} \mathbf{v}_M \\ -\bar{\mathbf{v}}_M \end{pmatrix} = (\sigma_M - \sigma_N) \begin{pmatrix} \mathbf{u}_M \\ -\bar{\mathbf{u}}_M \end{pmatrix}.$$

Proof. With the definition of N as in (4.6), we see that $V_N = \bar{V}_M$. Hence (4.4b) becomes

$$N\bar{\mathbf{v}}_M = \sigma_N\mathbf{u}_M.$$

The remainder of the proof is left as Exercise 6.3. \square

With this result, we may derive the previous SVD for e^{izL} .

COROLLARY 6.3.1.

$$e^{izL} = U\Sigma V^*,$$

where the component matrices are defined in (4.8).

Proof. V is made up of $2n$ columns \mathbf{v} , each of which must satisfy $e^{izL}\mathbf{v} = \sigma\mathbf{u}$ from (4.4b). But n of those columns are given by the first equality in (6.7), and the remainder are given by the second equality. Hence

$$V \propto \begin{pmatrix} V_M & V_M \\ \bar{V}_M & -\bar{V}_M \end{pmatrix},$$

where the $2^{-1/2}$ factor in (4.8a) assures the proper normalization. Note that the minus sign in front of the final entry assures that V is unitary. The corresponding U is described by

$$U \propto \begin{pmatrix} U_M & U_M \\ \bar{U}_M & -\bar{U}_M \end{pmatrix},$$

where the normalization factors and negative sign play the same role. The corresponding entries of Σ are $\sigma_M + \sigma_N$ for the first N columns and $\sigma_M - \sigma_N$ for the second n columns. Once one recalls from (4.7) that $\sigma_M - \sigma_N = (\sigma_M + \sigma_N)^{-1}$, the result is proved. \square

EXERCISE 6.1. *Complete the proof of Corollary 6.1.1.*

EXERCISE 6.2. *Complete the proof of Lemma 6.2.*

EXERCISE 6.3. *Complete the proof of Lemma 6.3 by performing the multiplications.*

The next section considers a slightly different configuration, where the details of the pump matrix K are made more explicit.

7. Two-Mode Case, Nonisotropic Propagation. We next consider one particular case of nonisotropic propagation, namely,

$$(7.1) \quad \frac{d}{dz} \begin{pmatrix} \mathbf{a}_s \\ \bar{\mathbf{a}}_i \end{pmatrix} = i \begin{pmatrix} \delta P & K \\ -\bar{K} & -\delta P \end{pmatrix} \begin{pmatrix} \mathbf{a}_s \\ \bar{\mathbf{a}}_i \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

so $L \in \mathcal{C}^{4 \times 4}$. Note that in this case, one of the two modes of both \mathbf{a}_s and \mathbf{a}_i is coupled directly to itself ($p_{11} = 1$), while the other is coupled negatively ($p_{22} = -1$). (This type of coupling is representative of birefringence [14], [23], [26], [27].)

In this case we can establish conditions under which signal amplification occurs by determining when the eigenvalues of L are either real or imaginary, again with an eye to determining when the amplification from the pump waves is maximized. By proving the following theorem about the eigenvectors, the desired result follows as a corollary.

THEOREM 7.1. *Let L be defined as in (7.1). Then L^2 has one eigenvector of the form*

$$\begin{pmatrix} \mathbf{z} \\ \mathbf{e}_1 \end{pmatrix}, \quad \mathbf{z}, \mathbf{e}_1 \in \mathcal{C}^2.$$

Proof.

$$L^2 = \begin{pmatrix} \delta P & K \\ -\bar{K} & -\delta P \end{pmatrix} \begin{pmatrix} \delta P & K \\ -\bar{K} & -\delta P \end{pmatrix} = \begin{pmatrix} \delta^2 P^2 - K\bar{K} & \delta(PK - KP) \\ \delta(P\bar{K} - \bar{K}P) & \delta^2 P^2 - \bar{K}K \end{pmatrix}.$$

$P^2 = I$, so the lower-right entry is Hermitian. Hence we define

$$(7.2) \quad H = \delta^2 I - \bar{K}K.$$

We also note that

$$\begin{aligned} PK - KP &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{pmatrix} - \begin{pmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} k_{11} & k_{12} \\ -k_{12} & -k_{22} \end{pmatrix} - \begin{pmatrix} k_{11} & -k_{12} \\ k_{12} & -k_{22} \end{pmatrix} = \begin{pmatrix} 0 & 2k_{12} \\ -2k_{12} & 0 \end{pmatrix}, \\ H^T &= \delta^2 I - K^T K^* = \delta^2 I - K\bar{K}, \\ P\bar{K} - \bar{K}P &= \overline{(PK - KP)} = \overline{(PK - KP)} = \begin{pmatrix} 0 & 2\bar{k}_{12} \\ -2\bar{k}_{12} & 0 \end{pmatrix}, \end{aligned}$$

so we rewrite L^2 as

$$L^2 = \begin{pmatrix} H^T & 2\delta k_{12} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ 2\delta \bar{k}_{12} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & H \end{pmatrix}.$$

Without loss of generality, for algebraic simplicity we redefine our eigenvector \mathbf{z}_{L^2} for L^2 as

$$\mathbf{z}_{L^2} = \begin{pmatrix} z_1 \\ z_2 \\ 2\delta \bar{k}_{12} \\ 0 \end{pmatrix}.$$

We now compute each row of $L^2 \mathbf{z}_{L^2} = \lambda \mathbf{z}_{L^2}$, starting with the third row,

$$(7.3a) \quad \begin{aligned} 2\delta \bar{k}_{12} z_2 + 2\delta \bar{k}_{12} h_{11} &= \lambda(2\delta \bar{k}_{12}), \\ z_2 &= \lambda - h_{11}, \end{aligned}$$

which gives us z_2 in terms of λ . Moving to the fourth row, we obtain

$$(7.3b) \quad \begin{aligned} -2\delta\bar{k}_{12}z_1 + 2\delta\bar{k}_{12}h_{21} &= 0, \\ z_1 &= h_{21}, \end{aligned}$$

which gives us z_1 . Using our results from (7.3) in the first row, we have

$$\begin{aligned} h_{11}z_1 + h_{21}z_2 &= \lambda z_1, \\ h_{11}h_{21} + h_{21}(\lambda - h_{11}) &= \lambda h_{21}, \end{aligned}$$

as required. Using our results from (7.3) in the second row yields

$$(7.4) \quad \begin{aligned} h_{12}z_1 + h_{22}z_2 - 2\delta k_{12}(2\delta\bar{k}_{12}) &= \lambda z_2, \\ h_{12}h_{21} + h_{22}(\lambda - h_{11}) - 4\delta^2|k_{12}|^2 &= \lambda(\lambda - h_{11}), \end{aligned}$$

which is a quadratic with at least one root λ , and generically has two roots. \square

It can be shown that Theorem 7.1 still holds true if one replaces \mathbf{e}_1 by \mathbf{e}_2 (see Exercise 7.1). This is how the other two eigenvectors are obtained.

Now we have the foundation to prove the following result about the eigenvalues.

COROLLARY 7.1.1. *Let L be defined as in (7.1). Then L has all real and imaginary eigenvalues if and only if*

$$(7.5) \quad d \equiv (h_{11} - h_{22})^2 + 4|h_{12}|^2 - 16\delta^2|k_{12}|^2 \geq 0.$$

Proof. If (7.5) is satisfied, then L^2 has real eigenvalues (see Exercise 7.2). Since $\lambda_{L^2} = \lambda_L^2$, we have that at least one (but generally two) eigenvalues of L must be either real or imaginary. But by the quartet structure, that forces all of them to be either real or imaginary, since $n = 2$. \square

Note that the proof provides only a condition on L which will lead to either no signal amplification (real eigenvalues) or pure signal amplification (imaginary eigenvalues). Further analysis of L^2 is then needed to determine which case is exhibited.

We conclude by examining one final case of physical interest. In degenerate FWM, a strong pump drives weak signal and idler sidebands (see the left-hand diagram in Figure 1.1). In this case, the coupling term K modeling the interaction between pump and sidebands is of the form [26]

$$(7.6) \quad K = \begin{pmatrix} \alpha p_x^2 & p_x p_y \\ p_x p_y & \alpha p_y^2 \end{pmatrix},$$

where p_x and p_y are (possibly complex) coupling coefficients in the x - and y -directions (orthogonal to propagation) and α is a real constant. In this case, we have

$$(7.7) \quad \begin{aligned} H &= \begin{pmatrix} \delta^2 - |p_x|^2(\alpha^2|p_x|^2 + |p_y|^2) & -\alpha\bar{p}_x p_y(|p_x|^2 + |p_y|^2) \\ -\alpha p_x \bar{p}_y(|p_x|^2 + |p_y|^2) & \delta^2 - |p_y|^2(|p_x|^2 + \alpha^2|p_y|^2) \end{pmatrix}, \\ d &= \alpha^2(|p_x|^2 + |p_y|^2)^2 [\alpha^2(|p_x|^2 - |p_y|^2)^2 + 4|p_x|^2|p_y|^2] - 16\delta^2|p_x|^2|p_y|^2. \end{aligned}$$

Note that the form of the matrix K as defined in (7.6) is not guaranteed to be diagonalizable.

PROPOSITION 7.2. *Let K take the form indicated in (7.6). Then K is diagonalizable and nontrivial if and only if the complex pump amplitudes p_x and p_y satisfy*

$$p_x p_y \neq \pm \frac{1}{2}i\alpha(p_x^2 - p_y^2).$$

Proof. Suppose that the condition is not satisfied. Then solving for the eigenvalue gives

$$\lambda = \frac{1}{2}\alpha(p_x^2 + p_y^2),$$

yielding

$$K - \lambda I = \frac{1}{2}\alpha(p_x^2 - p_y^2) \begin{pmatrix} 1 & \pm i \\ \pm i & -1 \end{pmatrix}.$$

This matrix has nondegenerate eigenvectors $(1, \pm i)^T$ unless $\alpha = 0$ or $p_x = \pm p_y$. Since both of these cases imply the trivial matrix $K = O$, the condition is sufficient. The proof in the other direction is left as Exercise 7.4. \square

EXERCISE 7.1. Show that Theorem 7.1 holds true if \mathbf{e}_1 is replaced by \mathbf{e}_2 .

EXERCISE 7.2. Show that if (7.5) is satisfied, L^2 has real eigenvalues.

EXERCISE 7.3. Verify the calculations in (7.7).

EXERCISE 7.4. Complete the proof of Proposition 7.2.

8. Conclusions. We have analyzed two complementary descriptions of linearly coupled envelope equations representing optical fields interacting through the optical Kerr nonlinearity present in optical fiber. The two descriptions are the differential system (2.4) obtained by linearizing the coupled nonlinear Schrödinger equations for resonantly interacting fields, referred to as the coupled-mode equations (CMEs), and the algebraic system (1.1b) obtained by solving the differential system, referred to as the input-output relations (IORs).

The CMEs are characterized by the matrix L in (2.5b). It was shown that the desirable characteristic of signal amplification was related to whether L had eigenvalues with negative imaginary part. This formed one of the key questions of this tutorial. L was shown to have a spectrum consisting of quartets $\{\lambda, \bar{\lambda}, -\lambda, -\bar{\lambda}\}$ of eigenvalues. This has the immediate consequence that an odd number of interacting (complex) modes will exhibit at least two purely imaginary or purely real eigenvalues, implying either a pure linear growth instability (desirable for signal amplification) or pure oscillatory dynamics, respectively.

The IORs are characterized by the matrix e^{izL} , which was shown to be symplectic. This allowed the identification of a simple singular value decomposition for the matrix and an alternative proof of the eigenvalue quartets. A second key goal was to draw any relationships between the spectral decomposition and the singular value decomposition for the system. We found only a few rudimentary results, but nothing with wide application. In fact, we found several counterexamples that illustrated that without key symmetry conditions, there are no simple relationships between them.

As a specific application of our results, we considered two particular two-mode cases. In both cases, the signal and idler modes interact only because of the pump-induced fiber nonlinearity. In the undepleted-pump approximation, this coupling is linear in the signal and idler (sideband) amplitudes. In the isotropic (nonbirefringent) case, the sideband polarization components have the same wavenumber, so the coupling term is δI . In this case (and, more generally, whenever AB is symmetric), the eigenvalues and eigenvectors of L can be constructed entirely from the singular vectors of the symmetric submatrix K . The eigenvalues are either real or purely imaginary, which again correspond to pure linear growth or oscillations.

In the nonisotropic (birefringent) case, the polarization components have different wavenumbers. If one transforms out the average wavenumber, the differences that

remain produce the δP term. In this case the eigenvalues can be shown to be real or pure imaginary if a certain criterion involving the matrix entries is satisfied.

Although the discussion here has focused on the application of (1.1a) to modeling parametric interactions in optical fiber, it is important to note that it arises in other areas. For instance, it arises in other optical contexts such as amplification, signal processing, and storage in engineered photonic structures. Outside the field of optics, a phase-conjugated term can be found in nonlinear evolution equations for Faraday oscillations in fluids [34] and for driven ferromagnets [6]. More broadly, nonnormal and non-self-adjoint operators are of interest in a wide variety of applications [42]. In each of these cases, a detailed understanding of the spectral properties considered here is critical to gaining physical insight into the problem.

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