Was Calculus Invented in India?

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Introduction

No. Calculus was not invented in India. But two hundred years before Newton or Leibniz, Indian astronomers came very close to creating what we would call calculus. Sometime before 1500, they had advanced to the point where they could apply ideas from both integral and differential calculus to derive the infinite series expansions of the sine, cosine, and arctangent functions:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} - \cdots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} - \cdots,$$

$$\arctan x = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} - \cdots.$$

Roy [**13**] and Katz [**7, 8**] have given excellent expositions of the Indian derivation of these infinite summations. I will give a slightly different explanation of how Indian astronomers obtained the sine and cosine expansions, with an emphasis on the succession of problems and insights that ultimately led to these series.

This story provides illuminations of calculus that may have pedagogical implications. The traditional introduction of calculus is as a collection of algebraic techniques that solve essentially geometric problems: calculation of areas and construction of tangents. This was not the case in India. There, ideas of calculus were discovered as solutions to essentially algebraic problems: evaluating sums and interpolating tables of sines.

Geometry was well-developed in pre-1500 India. As we will see, it played a role. But it was, at best, a bit player. The story of calculus in India shows us how calculus can emerge in the absence of the traditional geometric context. This story should also serve as a cautionary tale, for what did emerge was sterile. These mathematical discoveries led nowhere. Ultimately, they were forgotten, saved from oblivion only by modern scholars.

Greek origins of trigonometry

Trigonometry arose from, and for over fifteen hundred years was used exclusively for, the study of astronomy/astrology. Hipparchus of Nicæa (*ca.* 161–126 BC) is considered

the greatest astronomer of antiquity and the originator of trigonometry. Trigonomet was born in response to a scientific crisis. The Greek attempt to caste astronomy in th language of geometry was running up against the disturbing fact that the heavens a lop-sided. New tools were needed for analyzing astronomical phenomena.

Let me paint the background to this crisis. It begins with the assumption that th earth is stationary. While this was debated in early Greek science—does the earth g around the sun or the sun around the earth?—the simple fact that we perceive no sens of motion is a powerful indication that the earth does not move. In fact, when in th early seventeenth century it became clear that the earth revolves about the sun, it cr ated a tremendous problem for scientists: How to explain how this was possible? Ho could it be that we were spinning at thousands of miles per hour and hurtling throug space at even greater speeds without experiencing any of this? Surely if the earth di move, we would have been flung off long ago. Newton's great accomplishment in th *Principia* was to solve this problem. He created inertial mechanics for this purpos building it with the then nascent tools of calculus.

So we begin with a fixed and immovable earth. Above it is the great dome of th night sky, rotating once in every 24 hours. In far antiquity it was realized that th stars do not actually disappear during the day. They are present, but impossible to se against the glare of the sun. The position of the sun in this dome is not fixed. Durir the year, it travels in its own circle, called the *ecliptic*, through the constellations. Or can tell the season by locating the position of the sun in its annual journey around th great circle. This is what the zodiac does. The sign of the zodiac describes the locatic of the sun by pinpointing the constellation in which it is located (see the cover for medieval rendering of the zodiac).

Most stars are fixed in the rotating dome of the sky, but a few, called the *wande ers* or, in Greek, the *planetes* (hence our word planets), also move across the dor following this same ecliptic circle. If the position of the sun is so important in d termining seasons of heat and cold, rain and drought, it appears self-evident that th positions of the wanderers should have important—if more subtle—influences on o lives. Astronomy/astrology was born.

Aristotle, in the 4th century BC, inherited a world-view that saw the earth as th fixed center of the universe with the moon, sun, and planets embedded in concentri ethereal spheres that rotated with perfect regularity around us. It became the basis f a comprehensive world-view that was tight and consistent and would last for almo two millenia. But its first cracks appeared in less than two hundred years.

The four cardinal points of the great circle traversed by the sun mark the boun aries of the seasons: winter solstice, spring equinox, summer solstice, and autumn equinox. If the sun travels the ecliptic at constant speed, the four seasons should be equal length. They are not (see Figure 1). Winter solstice to spring equinox is a sho 89 days. Spring is almost 90 days. Summer, the longest season, is over $93\frac{1}{2}$ days. Ar fall comes close to 93 days. If, in fact, the sun moves at a constant speed, this can on mean that the earth is off-center. Hipparchus tackled the problem of calculating th position of the earth.

The basic problem of trigonometry as understood by Hipparchus and his conten poraries is the following: Given an arc of a circle, find the length of the chord th connects the endpoints of that arc (see Figure 2). This chord length depends on bo the length of the arc and the radius of the circle. For the Greeks, as for all scientis right through Newton, 90° was not the measure of a right angle, but of the distanc around one quarter of the circumference of a circle. Degrees were a measure of di tance. Given a circle of circumference 360°, it would be natural to take the radius be $360/2\pi = 57.2957795\ldots$ For greater accuracy, the circumference of this standa

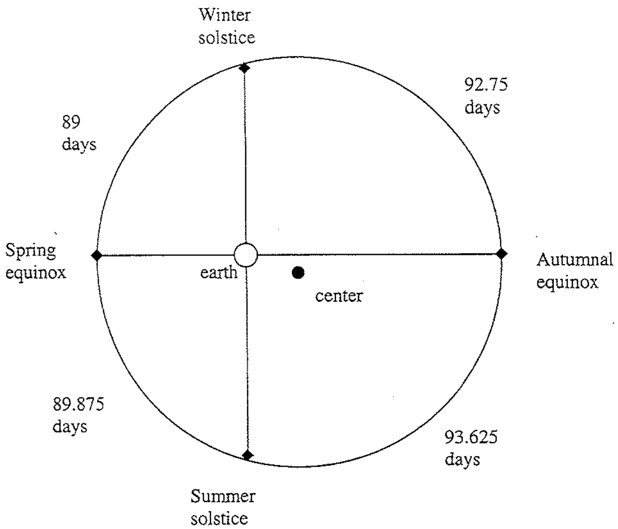

Figure 1. The unequal seasons, rounded to nearest 1/8 day.

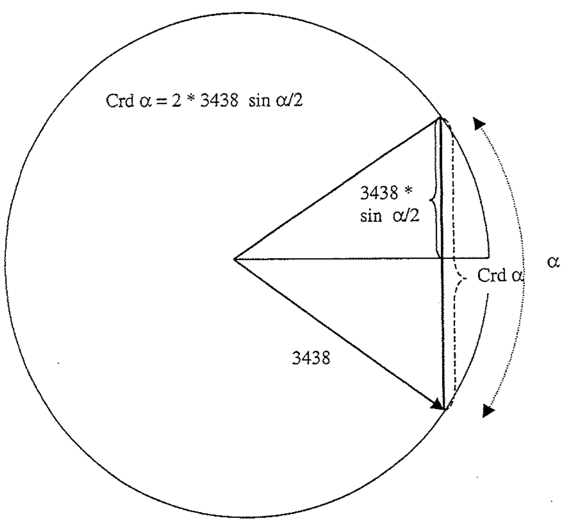

Figure 2. The relationship between crd α and $\sin\alpha$.

circle could be measured in minutes. The circumference is then 21,600 minutes a the radius is 3437.74677... It would become common in Indian trigonometry to u a radius of 3438. There is some evidence that Hipparchus, whose trigonometric tabl no longer survive, also may have used a radius of 3438.

Hipparchus was probably the first to construct a table of values of the length the chord for a given arc, what is sometimes called crd α. In modern trigonomet notation, the chord is twice the sine of half the angle, multiplied by the radius of t circle which we will take to be 3438 (see Figure 2):

$$\text{crd}\ \alpha = 3438 \cdot 2\sin(\alpha/2).$$

For the problem of the position of the earth, the arc from the winter solstice to t summer solstice is approximately 176°18′. Assuming that we know that crd 176°18′ 6872′, it follows that half of the chord is 3436′. We can now use the Pythagore theorem to find the distance from the center of the circle to this chord:

$$\text{distance} = \sqrt{3438^2 - 3436^2} \approx 117'.$$

This chord is 117 minutes, almost two full degrees, off center.

Over succeeding centuries, as astronomical observations became more accura the model for the movement of sun and planets became more complicated. Plan will seem to pause and reverse direction.[1] This was explained by putting small sphe inside each crystal ring, epicycles on which each planet would rotate around a po which itself traveled around the earth. Even with an off-centered earth, it was necess to vary the speed of the spheres. This was often accomplished by adding an equant

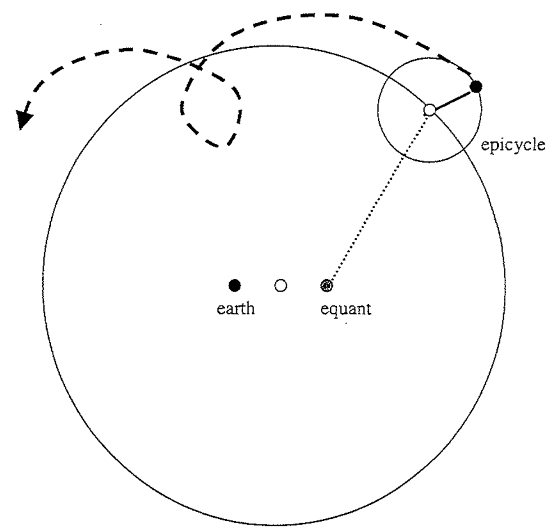

Figure 3. An epicycle combined with an equant. The planet circles the center of the epicy The center of the epicycle moves so that its angular velocity relative to the equant is const

[1]See `http://alpha.lasalle.edu/~smithsc/Astronomy/retrograd.html` for an illustration and planation of retrograde motion.

point from which the angular velocity of the center of the small circle appears constant (see Figure 3). All of the workings of this model relied on trigonometric calculations, and these calculations relied on an accurate table of chords.

By the end of the first century AD, Menelaus of Alexandria knew the formulas for the chords of sums and differences of angles, double and half angles. With these, he was able to construct an accurate table of chords. In the second century, Ptolemy, also of Alexandria, published his system of the heavens, including a table of chords for angles in increments of half a degree, equivalent to a table of sines in increments of a quarter-degree. It is important to our story to look at how this table was constructed. While it was given as a table of chords, I will explain it in terms of more familiar sines.

Beginning with the fact that $\sin 30^\circ = 1/2$ and using the half angle formula

$$\sin\alpha = \sqrt{\frac{1-\cos(2\alpha)}{2}},$$

one can calculate the sines of 15°, 7°30′, and 3°45′. Going back at least to Archimedes, it was known that

$$\sin 36^\circ = \frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{2}},$$

and so we get the sine of 18°. Using the sines of 15° and 18°and the difference of angles formula,

$$\sin(\alpha-\beta) \;=\; \sin\alpha\,\cos\beta - \cos\alpha\,\sin\beta,$$

we get the sine of 3°. Now we can calculate the sines of 1°30′ and 45′.

We are down to a very small angle. The Greeks knew that for very small angles, we have the approximation

$$\frac{\sin\alpha}{\sin\beta} \approx \frac{\alpha}{\beta}.$$

It follows that

$$\sin 1^\circ \approx \frac{4}{3}\,\sin\frac{3^\circ}{4}\,.$$

This is not a bad approximation. The error is of the same order of magnitude as that introduced by using 3438 as the radius of a circle of circumference 21600, less than 1 part in 10000. Two more iterations of the half angle formula, and we are down to the sine of 15′. Now we can use the sum and difference of angles formulas to fill in the missing values in the table.

Ptolemy's *Almagest* was the last great scientific achievement of the Græco-Roman world. Fortunately, India was just coming into its high classical period. Indian astronomers learned of the Greek accomplishments and began to incorporate them into their own science.[2] They did not stop with borrowing Greek ideas. They began to improve on what the Greeks had accomplished. Among their improvements would be conceptual breakthroughs that would allow them to reduce the errors to 1 part in 10^{12}.

[2] According to Neugebauer and Pingree [9], the Paulisa-siddhānta and the Romaka-siddhānta (4th century or earlier) are based on Greek astronomical works. Similarities in terminology, calculations, and choices of constants—as well as the names of these works—argue for the importation of Greek astronomical techniques.

Trigonometry in classical India

One of the first innovations was to work with the half-chord rather than the Greek chord, what was called the *ardha-jyā* or "half bowstring," eventually simplified to just *jyā* (bowstring) or *jīvā*. Islamic astronomers learned much of their trigonometry from India; Europe would learn it from North Africa. That is why today we use sines instead of chords.[3] But the greatest contribution to trigonometry to come out of India was the analysis of how to interpolate the tables of sines. From this would come the power series for the sine and cosine.

Āryabhaṭa, born in 476, analyzed a fourth century Sanskrit table of sines and described an interesting pattern when he took differences of consecutive entries, and then differences of those differences:

α	$3438\,\sin\alpha$	1st difference	2nd difference
3°45′	225		
7°30′	449	224	−2
11°15′	671	222	−3
15°	890	219	−4
18°45′	1105	215	−5
22°30′	1315	210	−5
26°15′	1520	205	−6
30°	1719	199	

Āryabhaṭa observed that these second differences are very close to the value in the second column divided by 225:

$$\big[\sin(x+225') - \sin(x)\big] - \big[\sin x - \sin(x-225')\big] \approx \frac{-\sin x}{225}.$$

Datta and Singh ([**3**, pp. 75–77]) argue that this could have been derived from the trigonometric identity

$$[\sin(x+\alpha) - \sin(x)] - [\sin x - \sin(x-\alpha)] = -\sin x\left(\frac{2\sin(\alpha/2)}{3438}\right)^2, \quad (1)$$

in which the argument of the sine is measured in minutes. This derivation is pure speculation, but it does illustrate how Āryabhaṭa's successors might have come to discover the second derivative formula for the sine. The sum of angles and half-angle formulas that are needed to derive (1) were certainly known by 1200 and probably long before that. Ever since the inception of trigonometry, it had been known that $2\sin(\alpha/2)/\alpha$ is approximately 1 for small values of α. When the argument of the sine function is measured in minutes, it follows from (1) that

$$\lim_{\alpha\to 0}\frac{\sin(x+\alpha) - 2\sin x + \sin(x-\alpha)}{\alpha^2} = \frac{-\sin x}{3438^2}\lim_{\alpha\to 0}\left(\frac{2\sin(\alpha/2)}{\alpha}\right)^2 = \frac{-\sin x}{3438^2}.$$

[3] According to Datta and Singh [3], Arab mathematicians used the term *jiba*, clearly derived from the Sanskrit *jīvā*. When Gherardo of Cremona (*ca.* 1150) translated this into Latin, he misread it as *jaib* which is Arabic for "bosom" or "bay," and translated it as *sinus* from which we get *sine*.

By 665, Brahmagupta of Bhillamāla (modern Bhinmal) in Rajasthan had found the formula that showed how to use the second differences to approximate interpolated values. We assume that we want to find the value of $\sin(x+\epsilon)$ where x is the nearest angle for which we know $\sin x$. We also assume that α is the common difference between angles in our table, so that we also know the sines of $x+\alpha$ and $x-\alpha$. These can be used to approximate the first and second derivatives of $\sin x$:

$$\frac{d}{dx}\sin x \approx \frac{\sin(x+\alpha)-\sin(x-\alpha)}{2\alpha},$$

$$\frac{d^2}{dx^2}\sin x \approx \frac{\sin(x+\alpha)-2\sin x+\sin(x-\alpha)}{\alpha^2}.$$

Brahmagupta stated that

$$\sin(x+\epsilon)\approx \sin x+\epsilon\,\frac{\sin(x+\alpha)-\sin(x-\alpha)}{2\alpha} + \frac{\epsilon^2}{2}\,\frac{\sin(x+\alpha)-2\sin x+\sin(x-\alpha)}{\alpha^2}.$$

It is worth noting that this formula is valid no matter what units—degrees, minutes, or radians—we use to measure ϵ and α. We do, however, have to use the same units for both.

What Brahmagupta had discovered is the quadratic case of the Newton interpolation formula.[4] The right side is the unique quadratic polynomial that agrees with the sine at $x-\alpha$, x, and $x+\alpha$. Note that if α and ϵ are measured in radians and we take the limit as $\alpha \to 0$, then we get the familiar Taylor polynomial in ϵ:

$$\sin(x+\epsilon)\approx \sin x+\epsilon\cos x-\frac{\epsilon^2}{2}\sin x.$$

In the early ninth century, Govindasvāmin of Kerala showed how to extend Brahmagupta's quadratic formula to interpolation formulas for higher powers.

By the twelfth century, Bhāskara II was using the fact that the first difference of the sine, $\sin(x+\epsilon)-\sin x$, is close to $\epsilon\cos x$ in the sense that their ratio approaches 1 as ϵ approaches 0. He also used $(-\sin x)\epsilon^2$ to approximate the second difference of the sine. Around 1400 in a commentary on the work of Govindasvāmin, Parameśvara used the limits of the first, second, and third differences to give a cubic approximation for $\sin(x+\epsilon)$ when $\sin x$ is known:

$$\sin(x+\epsilon)=\sin x+\frac{\epsilon}{R}\cos x-\frac{\epsilon^2}{2R^2}\sin x-\frac{\epsilon^3}{4R^3}\cos x,$$

where the arguments of the trigonometric functions are measured in units equal to R^{-1} radians. This formula is not quite correct. The last denominator should be $6R^3$. But the Indian astronomers were on their way to the general MacLaurin expansions of the sine and cosine.

The exact date and attribution of the series for sine, cosine, and arctangent are uncertain. The earliest unquestioned appearance of these series is in the *Yuktibhāṣā* written by Jyeṣṭhadeva in the early 1500s. Jyeṣṭhadeva based his work on the *Tantrasaṃgraha*

[4] Newton's interpolation formula appears in his *Principia Mathematica*. Brook Taylor used it to derive the Taylor series.

of Nīlakaṇṭha, written in 1501. A commentary on the *Tantrasaṃgraha* by one (Jyeṣṭhadeva's students, the *Tantrasaṃgraha-vyākhyā*, written prior to 1550, has l(Rajagopal and Rangachari [11] to argue that Nīlakaṇṭha was familiar with these seri(and that they were part of the oral tradition that accompanied his work. The series f(the sine does appear in one of Nīlakaṇṭha's later works, the *Āryabhaṭīya-bhāṣya*, wri ten prior to 1545, where he attributes it to Mādhava who lived approximately 134(1425. There is additional evidence from the results that Mādhava is known to ha(authored that he probably did know these series.[5] What this all means is that the da of discovery of these series cannot be pinned down any more accurately than aft 1350 and before 1550, with evidence suggesting the earlier rather than the later pa of this window. In any event, they were discovered in India well over a century befo their rediscovery in Europe.

The power series expansion for sine

Up to this point, I have translated the Indian formulas into more familiar sines a(cosines, but to do proper justice to the Indian derivation of the sine series, I need state and follow the proof in something closer to the original notation. I will use jyā and koj α (for kotijyā) to denote, respectively, the half-chord of the arclength α a(the half-chord of the complementary angle. Note that these are also dependent on t(radius, R. If the sine and cosine are functions of angles measured in radians, then

$$\text{jyā}\,\alpha = R\sin(\alpha/R),$$

$$\text{koj}\,\alpha = R\cos(\alpha/R).$$

We will present Jyeṣṭhadeva's proof that

$$\text{jyā}\,\alpha=\alpha-\frac{\alpha^3}{R^2\,3!}+\frac{\alpha^5}{R^4\,5!}-\frac{\alpha^7}{R^6\,7!}+\cdots.$$

The first step is to find the limiting formulas for the first difference of the jyā a(kotijyā. In Figure 4, we let $\widehat{PX}$ be the arclength α and $\widehat{PR}$ be $\Delta\alpha$, the change in α. T(problem is to estimate $RS=\Delta(\text{jyā}\,\alpha)=\text{jyā}(\alpha+\Delta\alpha)-\text{jyā}\,\alpha$ and $PS=\Delta(\text{koj}$ (We mark Q, the midpoint of arc $\widehat{PR}$, and note that OQ is the perpendicular bisec(of chord PR.

For a small change in arclength, the chord PR is a very good approximation to t(arc $\widehat{PR}$, and so we will not distinguish between them. Also, BQ equals jyā$(\alpha+\frac{1}{2}\Delta$(which we will identify with jyā α. Similarly, we treat OB as equal to koj α. Trian(RSP is similar to triangle OBQ, and therefore

$$\frac{RS}{PR}=\frac{OB}{OQ}\Longrightarrow \Delta(\text{jyā}\,\alpha)=\frac{(\Delta\alpha)\,\text{koj}\,\alpha}{R}, \quad ($$

$$\frac{PS}{PR}=\frac{BQ}{OQ}\Longrightarrow \Delta(\text{koj}\,\alpha)=\frac{-(\Delta\alpha)\,\text{jyā}\,\alpha}{R}. \quad ($$

In modern terms, Jyeṣṭhadeva's next step is to observe that

$$\sin\alpha=\int_0^\alpha \cos x\,dx, \quad\text{and}\quad \cos\alpha = 1-\int_0^\alpha \sin x\,dx.$$

[5] See the analyses by Pingree [10] and Sarma [15].

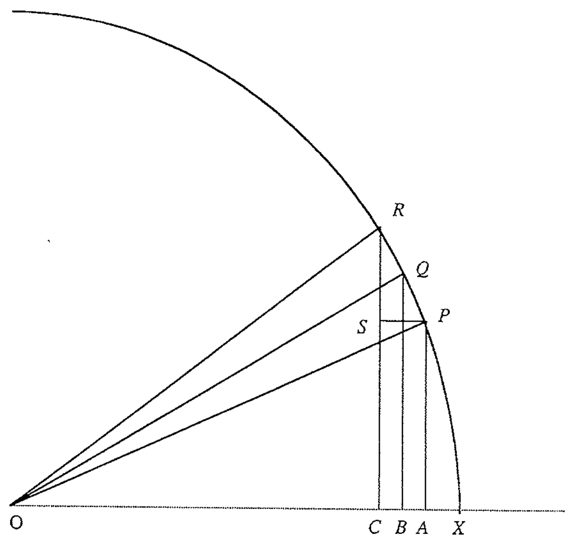

Figure 4. $RS = \Delta(\text{jyā}\,\alpha)$ and $PS = \Delta(\text{koj}\,\alpha)$.

Using these equalities, a polynomial approximation to the sine can be turned into an approximation of the cosine with degree one higher. A polynomial approximation of the cosine can be turned into an approximation of the sine with degree one higher. We then iterate this process to generate the infinite series.

Let me put the preceding paragraph into notation that is still modern but closer to the spirit of Jyeṣṭhadeva's construction. He sets $\alpha = n\,\Delta\alpha$ and observes that the sum of small differences is equal to the large difference:

$$\text{koj}\,\alpha - \text{koj}\,0 = \sum_{i=0}^{n-1} \Delta(\text{koj}\,(i\,\Delta\alpha)).$$

He uses (3) and the approximation $\text{jyā}\,\alpha \approx \alpha$ to simplify this:

$$\begin{aligned} \text{koj}\,\alpha - \text{koj}\,0 &= \sum_{i=0}^{n-1} \frac{-(\Delta\alpha)\text{jyā}\,(i\,\Delta\alpha)}{R} \\ &= \frac{-(\Delta\alpha)^2}{R}\sum_{i=0}^{n-1} i \\ &= \frac{-(\Delta\alpha)^2(n^2-n)}{2R}. \end{aligned} \tag{4}$$

We know that $\text{koj}\,0 = R$, $n\Delta\alpha = \alpha$, and $(\Delta\alpha)^2 n$ can be made arbitrarily small by taking $\Delta\alpha$ sufficiently small. Taken with (4), this implies that

$$\text{koj}\,\alpha \approx R - \frac{\alpha^2}{2R}. \tag{5}$$

We now use this approximation and the result given in (4) to improve the approx mation to $\text{jyā}\,\alpha$:

$$\begin{aligned} \text{jyā}\,\alpha - \text{jyā}\,0 &= \sum_{i=0}^{n-1} \Delta(\text{jyā}\,(i\,\Delta\alpha)) \\ &= \sum_{i=0}^{n-1} \frac{\Delta\alpha}{R}\left(R - \frac{(i\,\Delta\alpha)^2}{2R}\right) \\ &= n\,\Delta\alpha - \frac{(\Delta\alpha)^3}{2R^2}\sum_{i=0}^{n-1} i^2. \end{aligned} \quad (($$

We use the fact that $\sum_{i=0}^{n-1} i^2$ is $n^3/3$ plus lower order terms to get the improved a proximation:

$$\text{jyā}\,\alpha \approx \alpha - \frac{\alpha^3}{2\cdot 3\,R^2}. \quad (($$

In the next iteration we need to know that $\sum_{i=0}^{n-1} i^3$ is $n^4/4$ plus lower order term For the general iterative step, we need to know that

$$\sum_{i=0}^{n-1} i^k = \frac{n^{k+1}}{k+1} + \text{lower order terms.} \quad ($$

Today we recognize that $\sum_{i=0}^{n-1}(i/n)^k(1/n)$ is a Riemann sum for $\int_0^1 x^k\,dx$. In oth words, what we need to know is that

$$\int_0^\alpha x^k\,dx = \frac{\alpha^{k+1}}{k+1}.$$

Jyeṣṭhadeva's argument for (8) is given by Roy [**13**]. Katz [**8**] describes al-Haytham derivation of (8) for $k \le 4$, an approach that is easily extended to any value of al-Haytham lived in eleventh century Egypt, but knowledge of his results may ha traveled to India. In fact, this asymptotic estimate for the summation of the kth powe became widely known in the Middle East and India before the fifteenth century. I sh describe the approach used by Nārāyaṇa in his *Gaṇitakaumudī*, written in 1356.

Nārāyaṇa built on earlier observations that

$$\sum_{i=1}^{n}\binom{i}{1} = \binom{n+1}{2},$$

$$\sum_{i=1}^{n}\binom{i+1}{2} = \binom{n+2}{3},$$

$$\sum_{i=1}^{n}\binom{i+2}{3} = \binom{n+3}{4}.$$

The first two of these are ancient and can be found in both Greek and early Jain mat ematics. They lend themselves to geometric proofs. Nārāyaṇa's greatest accomplis ment was to view these not as geometric, but as formulaic or algebraic (though, course, he did not have the advantages of our notation). He thought of them as iterated sums. This suggested the following generalization:

$$\sum_{i=1}^{n} \binom{i+k-1}{k} = \binom{n+k}{k+1}. \tag{9}$$

Each iterated sum, $\binom{n+k-1}{k}$, is equal to

$$\frac{n(n+1)(n+2)\cdots(n+k-1)}{k!} = \frac{n^k}{k!} + \text{lower order terms.}$$

It follows from (9) that

$$\sum_{i=1}^{n} \left(\frac{i^k}{k!} + \text{lower order terms}\right) = \frac{n^{k+1}}{(k+1)!} + \text{lower order terms,}$$

which implies (8). It is worth noting that Nārāyaṇa also showed how to use equation (9) to find sums of other specific polynomials in i by first expressing the polynomial as a linear combination of these binomial coefficients.

Conclusion

There is no evidence that the Indian work on series was known beyond India, or even outside Kerala, until the nineteenth century. Gold and Pingree assert [**4**] that by the time these series were rediscovered in Europe, they had, for all practical purposes, been lost to India. The expansions of the sine, cosine, and arc tangent had been passed down through several generations of disciples, but they remained sterile observations for which no one could find much use.

No. Calculus was not discovered in India. I am left wondering how much important mathematics is today known but not yet discovered, passed among a coterie of tightly knit disciples as an intriguing yet seemingly useless insight, lacking the context, the fertilizing connections, that would enable it to blossom and produce its fruit.

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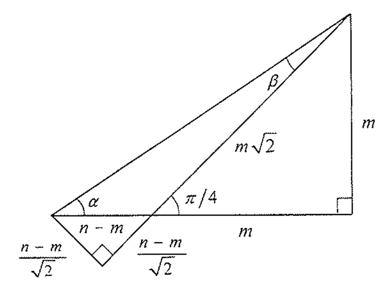
Mathematics Without Words

Geoffrey A. Kandall demonstrates that, when $0 < m < n$,

$$\tan^{-1}\left(\frac{m}{n}\right) + \tan^{-1}\left(\frac{n-m}{n+m}\right) = \frac{\pi}{4}:$$



$$\alpha = \tan^{-1}\left(\frac{m}{n}\right), \quad \beta = \tan^{-1}\left(\frac{(n-m)/\sqrt{2}}{(n-m)/\sqrt{2} + m\sqrt{2}}\right) = \tan^{-1}\left(\frac{n-m}{n+m}\right),$$

$$\alpha + \beta = \pi/4.$$