Homework 5, Due Monday, Feb 26

This homework must be done individually. Remember to follow Math department's guidelines for homework. Please write your solutions neatly. Typesetting in LaTeX is appreciated and encouraged. **Always show your work and justify your answers.**

1. Consider the principal ideal $\langle 2 \rangle$.
   (a) Show that as an ideal of $\mathbb{Z}$, $\langle 2 \rangle$ is maximal.
   (b) Show that as an ideal of $\mathbb{Z}[x]$, $\langle 2 \rangle$ is not maximal.
2. Consider $I = \{(3a, b) : a, b \in \mathbb{Z}\}$.
   (a) Show that I is an ideal of $\mathbb{Z} \times \mathbb{Z}$.
   (b) Determine whether I is a maximal ideal of $\mathbb{Z} \times \mathbb{Z}$.
3. Consider $I_0 = \{p(x) \in \mathbb{Z}[x] : p(0) = 0\}$.
   (a) Show that I_0 is an ideal of $\mathbb{Z}[x]$.
   (b) Show that for any $n \in \mathbb{Z}^+$, $\exists$ a sequence of strictly increasing ideals such that $I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_n \subset \mathbb{Z}[x]$.
   (c) Show that for any $n \in \mathbb{Z}^+$, $\exists$ a sequence of strictly decreasing ideals of $\mathbb{Z}[x]$ such that $I_0 \supset I_1 \supset I_2 \supset \cdots \supset I_n$.
   (d) Express each ideal I_k in part (c) [Not in part (b)] as a principle ideal.
4. Consider the quotient ring $\mathbb{Z}_3[x]/\langle x^2 + x + 1 \rangle$. Is it a field? Is it an integral domain? Is it a commutative ring with unity? Justify your claims.