Homework 10, Due Wed, Apr 25

This homework must be done individually. Remember to follow Math department's guidelines for homework. Please write your solutions neatly. Typesetting in LaTeX is appreciated and encouraged. **Always show your work and justify your answers.**

1. The set (field) of algebraic numbers is a central topic of study in the branch of modern mathematics called *algebraic number theory*. Let's prove a couple of basic facts about algebraic numbers.
   (a) Show that the square root of an algebraic number is also an algebraic number.
   (b) Show that Gaussian integers $\mathbb{Z}[i]$ is a subset of the field of algebraic numbers. (Hint: If $\alpha \in \mathbb{C}$ is a root of a $p(x) \in \mathbb{R}[x]$, then so is its complex conjugate)
2. Let E be an extension field of F such that $[E : F] = p$ for a prime number p. Let $\alpha \in E$. What are the possible values of $[F(\alpha) : F]$? How exactly is $F(\alpha)$ related to E and F in each case? Explain.
3. Let E be an extension of F and let $\alpha, \beta \in E$ be algebraic over F of degree n_1 and n_2 respectively where n_1 and n_2 are relatively prime. Determine $[F(\alpha, \beta) : F]$. Justify your answer.
4. Show that $\mathbb{Q}(\sqrt{5}, \sqrt[3]{5}) = \mathbb{Q}(\sqrt[6]{5})$.
5. Let F be a field and $f(x), g(x)$ be irreducible over F such that $\gcd(\deg(f(x)), \deg(g(x)) = 1$. Let α be a root of $f(x)$ in some extension of F. Show that $g(x)$ is irreducible over $F(\alpha)$.
6. We proved that every finite extension is an algebraic extension. The converse, however, is not necessarily true. Give an example of an algebraic extension that is not a finite extension.
7. Do Problem 35 on page 293.