# Homework 7, Due Monday, Nov 3

This homework must be done individually. Remember to follow Math department's guidelines for homework, and the course policy on the use of AI tools. Please write your solutions neatly. Typesetting in LaTeX is appreciated and encouraged. **Always show your work and justify your answers.**

1. Let $G = \langle \mathbb{C}^*, \cdot \rangle$. Show that the map $\phi : G \to G$ defined by $\phi(x) = x^4$ is a homomorphism and determine its kernel. Determine the kernel of the map $\phi(x) = x^n$ for any positive integer $n$.

2. For a pair of positive integer $m$ and $n$ define $\phi : \mathbb{Z} \to \mathbb{Z}_m \times \mathbb{Z}_n$ by $x \to (x \mod m, x \mod n)$. Show that this map is a homomorphism and find its kernel. Express the kernel as a familiar subgroup of $\mathbb{Z}$. What is the kernel in the case $\gcd(m, n) = 1$?

3. Let $\phi : G \to G'$ be a group homomorphism where $|G|$ is a prime. Show that $\phi$ must be either the trivial homomorphism or must be one-to-one.

4. Let $\phi$ be a homomorphism from $\mathbb{Z}_{40}^*$ to $\mathbb{Z}_{40}^*$ with $\ker(\phi) = \{1, 9, 17, 33\}$. If $\phi(3) = 11$, find all elements of $\mathbb{Z}_{40}^*$ that map to 11.

5. Let $G$ be a group and let $h, k \in G$. Define $\phi : \mathbb{Z} \times \mathbb{Z} \to G$ by $\phi(m, n) = h^m k^n$. Find a necessary and sufficient condition for $\phi$ to be a homomorphism. Of course, justify your answer.