Homework on Section 6

Due Monday, Sep 29

This homework must be done individually without the help of any outside sources or AI tools. Remember to follow Math department's guidelines for healthy collaboration on homework, and the course policy on the use of AI tools. Please write your solutions neatly. Typesetting in LaTeX is appreciated and encouraged.

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1. Let G be a group and let $g \in G$ be such that $|g^5| = 12$. What are the possibilities for $|g|$? If $|a^4| = 12$, then what are the possibilities for $|a|$? Be as accurate and specific as possible about the possibilities.
2. Let G be a group and let $x, y \in G$ be such that $|xy|$ is finite. Show that $|xy| = |yx|$.
3. Let G be a group and let $x, y \in G$ be such that $|x| = m$ and $|y| = n$. Assume that x and y commute, i.e., $xy = yx$. Prove that
   (a) $|xy|$ divides $\operatorname{lcm}\{m, n\}$, the least common multiple of m and n.
   (b) Give an example to show that $|xy|$ can be equal to $\operatorname{lcm}\{m, n\}$.
   (c) Give an example to show that $|xy|$ can be less than $\operatorname{lcm}\{m, n\}$.
4. Let p be a prime and let n be a positive integer. If x is an element of the group G such that $x^{p^n} = 1$ then what are the possibilities for $|x|$? Determine all of the possibilities.
5. Let p be a prime number. Determine all subgroups of $\mathbb{Z}_p$. Justify your answer.
6. Recall the definition of the discrete logarithm $L_\alpha(\beta)$ from the handout. Show that the discrete logarithm satisfies the familiar property of the logarithmic function: $L_\alpha(\beta_1\beta_2) = L_\alpha(\beta_1) + L_\alpha(\beta_2)$