Foundations, Gift 5

Due Monday, October 7

LaTeXis required on this homework. Type your solutions in LaTeXand submit the pdf of your document to Moodle by 10 am on Monday, October 7.

1. Let $m, n \in \mathbb{Z}$. We say that "m divides n ", denoted by $m \mid n$, if there exists $k \in \mathbb{Z}$ such that $n = m \cdot k$. Prove that for all integers $n \geq 0$, $6 \mid (n^3 - n)$.

2. Let x be a real number such that $x + \dfrac{1}{x}$ is an integer. Show that $x^n + \dfrac{1}{x^n}$ is also an integer for every natural number n.

3. Let a_n be defined (recursively) as : $a_1 = 1, \quad a_{n+1} = \sqrt{3 + a_n}$ for $n \geq 1$. Show that
   - a_n is bounded from above by 3, that is, $a_n \leq 3$ for all $n \in \mathbb{N}$.
   - a_n is increasing, i.e., $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$.
   - Conclude that a_n is convergent.[1] Find its limit.

4. The Fibonacci sequence [2] is the sequence of numbers defined by $f_1 = 1, f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 3$. Prove that $\left(\dfrac{3}{2}\right)^{n-2} \leq f_n \leq 2^n$ for all $n \in \mathbb{N}$.

[1] This uses the following Theorem from Calculus: "If a sequences is monotone and bounded, then it has a limit." You are allowed to use this theorem without having to prove it.

[2] This is one of the many examples of misleadingly named mathematical objects. Do a little bit of research to find out why this is misleading naming. Do you know of other such examples in mathematics?