Homework 5, Due Monday, Oct 16

This homework must be done individually. Remember to follow Math department's guidelines for homework. Please write your solutions neatly. Typesetting in LaTeX is appreciated and encouraged. **Always show your work and justify your answers.**

1. Show that if $G_1 \times G_2$ is a cyclic group then both G_1 and G_2 are cyclic. Is the converse of this statement true as well?

2. Let $G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : a, b \in \mathbb{Z}_3 \right\}$

   Show that G is an abelian group with the usual matrix multiplication (except that addition and multiplication will be done mod 3). What is the size of G? What familiar group is it isomorphic to?

3. Let $G = \{ax^2 + bx + c : a, b, c \in \mathbb{Z}_5\}$. Define addition on G as the usual addition of polynomials except that you add the coefficients modulo 5. What is the size of G? What familiar group is it isomorphic to?

4. A DNA molecule is composed of 2 long strands in the form of a double helix. Each strand is made up of strings of 4 nitrogen bases adenine (A), thymine (T), guanine (G), and cytosine (C). Each base on one strand binds to a complementary base on the other strand. Adenine always bound to thymine, and guanine is always bound to cytosine ($A \leftrightarrow T$ and $G \leftrightarrow C$). We can represent the genetic code using the elements of $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \cdots \times \mathbb{Z}_4$ where we omit paranthesis and commas using the identification $A \leftrightarrow 0$, $T \leftrightarrow 2$, $G \leftrightarrow 1$, and $C \leftrightarrow 3$. Thus the DNA segment $GTTCCA$ and its complement $CAAGGT$ are denoted by 122330 and 300112.

   (a) What arithmetic operation in $\mathbb{Z}_4$ corresponds to interchanging pairs of nitrogen bases? That is, given a DNA segment $a_1 a_2 \cdots a_n$ represented as an element of $(\mathbb{Z}_4)^n$, determine its complementary segment.

   (b) Find a way to express a DNA string of length n as an element of $(\mathbb{Z}_2 \times \mathbb{Z}_2)^n = (\mathbb{Z}_2)^{2n}$. Make sure that your representation preserves the complementary base pairing property.