Homework 1 on Sections 1 and 2

Due Monday, Sep 4

This homework must be done individually. Remember to follow Math department's guidelines for homework. Please write your solutions neatly. Typesetting in LaTeX is highly appreciated and encouraged.

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1. Do problems 32 and 42 in section 1 of the textbook (pages 18 and 19).

2. Let A be a finite alphabet and let A^* denote the set of all sequences (or words) of finite lengths (including length 0) over A. Define an operation called concatenation defined on A^* as follows. For $\mathbf{a} = a_1a_2\dots a_n$ and $\mathbf{b} = b_1b_2\dots b_m$ in A^*, their concatenation is defined as $\mathbf{ab} = a_1a_2\dots a_nb_1b_2\dots b_m$. Determine if this operation is associative and commutative? Does it have an identity element?

3. Prove that if $\langle G, * \rangle$ is a group such that $x * x = e$ for every $x \in G$, then G is an abelian (commutative) group. Is the converse of this statement true as well?

4. Let $G = \mathbb{Q}[\sqrt{2}] := \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$. Show that G is a group under usual addition of real numbers. We say that G is a subgroup of R and denote this by $\langle G, + \rangle \leq \langle \mathbb{R}, + \rangle$.

5. Given $(a, b) \in \mathbb{R}^2$, define $T_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2$ by $T_{a,b}(x, y) = (x + a, y + b)$. Consider $G := \{T_{a,b} : a, b \in \mathbb{R}\}$ with the binary operation $\circ$ of function composition. Is $\langle G, \circ \rangle$ a group? Is it a commutative group? What is the geometric interpretation of the map $T_{a,b}$? What is the geometric explanation for $\circ$ being commutative or not?

6. Consider the matrices $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ These matrices are called Pauli matrices and they play a fundamental role in quantum computation. They represent the most commonly used single-cubit transformations. More specifically, I is the identity transformation, X is negation (interchanging the qubits $|0\rangle$ and $|1\rangle$ that form the standard/computational basis), Z changes the relative phase of a superpoistion in the standard basis, $Y = ZX$ is a combination of negation and phase change. Consider the set $G = \{\pm I, \pm X, \pm Y, \pm Z\}$ together with the usual matrix multiplication. Construct the operation table and show that this forms a group. This group is called the Pauli group. You can take it granted that matrix multiplication is associative. Is the Pauli group commutative?