$$y_2^*(x) = (n+1)!\sum_{k=0}^{\infty} x^{n+1+k}/(n+1+k)! = (n+1)!\sum_{j=n+1}^{\infty} x^j/j! = (n+1)!\left(e^x - \sum_{j=0}^{n} x^j/j!\right)$$
$$= (n+1)!\,e^x - (n+1)!\,y_n(x).$$

Thus, we have $y(x) = e^x$ and $y_n(x)$ as solutions of (1.1).

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COMMUTATORS AND THE COMMUTATOR SUBGROUP

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The commutator subgroup G' of a group G is generated by commutators, elements of the form $[x, y] = x^{-1}y^{-1}xy$. As is quite well known, not every element of G' need be a commutator. What is perhaps less well known is a convenient source of finite groups which are examples of this phenomenon. The purpose of this note is to provide such a source. (Other examples are described in **[1]**, **[2]** and **[3]**.) The method given here can be used to construct both solvable and nonsolvable groups and even yields examples which are perfect, that is $G' = G$. The author is unaware, however, of any nonabelian *simple* group which contains a noncommutator.

Our examples will be wreath products and we begin with a description of these groups. Let U and H be any groups. Their wreath product $G = U \wr H$ has as normal subgroup the group B of all functions $f: H \to U$. Multiplication in B is pointwise. Also $H \subseteq G$ and $G = BH$ (and of course $B \cap H = 1$). Finally, to complete the description of G we have for $f \in B$ and $h \in H$ that $h^{-1}fh = f^h \in B$ with $f^h(x) = f(xh^{-1})$ for $x \in H$. We refer to B as the *base group* of the wreath product.

THEOREM. *Let U and H be finite groups with U abelian and H nonabelian. Let $G = U \wr H$. Then G' contains a noncommutator if*

$$(*) \qquad \sum_{A \in \mathcal{A}} \left(\frac{1}{|U|}\right)^{|H:A|} \leqq \frac{1}{|U|},$$

where $\mathcal{A}$ is the set of maximal abelian subgroups of H. In particular, this condition holds whenever $|U| \geqq |\mathcal{A}|$.

Actually, as can be seen from the proof, somewhat weaker conditions that (*) suffice although they are hard to state cleanly. In fact, if H is nonabelian of order 6, we can take $|U| = 2$. Although (*) is not satisfied, nevertheless the resulting group G of order $2^7 \cdot 3$ is an example where G' contains a noncommutator.

LEMMA 1. *Let G be a group with abelian $A \lhd G$ and suppose $G = AH$ with $A \cap H = 1$. If $[x, y] \in A$ with $x, y \in G$, then $[x, y] \in [A, K]$ for some abelian $K \subseteq H$.*

Proof. Write $x = ah$ and $y = bk$ with $a, b \in A$ and $h, k \in H$. Since $[x, y] \in A$, the images of x and y in G/A commute. Since these are also the images of h and k, it follows that $[h, k] \in A$. Since also $[h, k] \in H$, we have $[h, k] = 1$ and $K = \langle h, k \rangle$ is abelian.

Now $[A, K]$, the group generated by commutators of elements of A with elements of K, is normal in AK. The images of A and K in $AK/[A, K]$ are abelian and centralize each other, and hence $AK/[A, K]$ is abelian. Thus $[x, y] \in (AK)' \subseteq [A, K]$. ∎

LEMMA 2. *Let $G = U \wr H$ where U is abelian and G is finite. Let B be the base group of G and let $K \subseteq H$. Then $|[B, K]| = |U|^{|H| - |H:K|}$.*

Proof. Let T be a set of representatives for the left cosets of K in H. For each $t \in T$, define $\sigma_t : B \to U$ by $\sigma_t(f) = \Pi_{k \in K} f(tk)$. Then σ_t is a homomorphism and $\sigma_t(f^k) = \sigma_t(f)$ for $f \in B$ and $k \in K$.

Let $C = \bigcap_{t \in T} \ker \sigma_t$. Then $|C| = |U|^{|H| - |H:K|}$ since any f in C may be specified arbitrarily on all but one element in each coset. We claim that $[B, K] = C$.

To show that $[B, K] \subseteq C$, let $f \in B$. Then $[f, k] = f^{-1} f^k$ and $\sigma_t([f, k]) = \sigma_t(f^{-1}) \sigma_t(f^k) = 1$. Thus $[f, k] \in C$ and hence $[B, K]$, the group generated by all $[f, k]$, is contained in C.

To show that $C \subseteq [B, K]$, let $\tau : B \to B/[B, K]$ be the canonical homomorphism and note that $\tau(f^k) = \tau(f)$ for $f \in B$. Let $c \in C$ and $k \in K$ and define $c_k \in B$ by

$$c_k(x) = \begin{cases} c(x) & \text{if } xk \in T \\ 1 & \text{if } xk \notin T. \end{cases}$$

It follows that $c = \Pi_{k \in K} c_k$. Let $b = \Pi_{k \in K} (c_k)^k$. Since $\tau(f) = \tau(f^k)$ we have $\tau(c) = \tau(b)$. We claim that $b = 1$ and thus $\tau(c) = 1$ and $C \subseteq [B, K]$. We compute $b(x)$ for $x \in H$. If $x \notin T$, then $(c_k)^k(x) = c_k(xk^{-1}) = 1$ for all k and $b(x) = 1$. If $x \in T$, then

$$(c_k)^k(x) = c_k(xk^{-1}) = c(xk^{-1})$$

and so $b(x) = \Pi_k c(xk^{-1}) = \sigma_x(c) = 1$. The proof is complete. ∎

Proof of Theorem. Let B be the base group of $G = U \wr H$. Then $[B, H] \subseteq G'$ and $|[B, H]| = |U|^{|H| - 1}$ by Lemma 2. If every element of $[B, H]$ is a commutator, then $[B, H] = \bigcup_{A \in \mathscr{A}} [B, A]$ by Lemma 1. Since $|[B, A]| = |U|^{|H| - |H:A|}$, this forces

$$\sum_{A \in \mathscr{A}} |U|^{|H| - |H:A|} > |U|^{|H| - 1}$$

and thus

$$\sum_{A \in \mathscr{A}} \left(\frac{1}{|U|} \right)^{|H:A|} > \left(\frac{1}{|U|} \right)$$

and the first statement is proved. The second statement follows since $|H:A| \geqq 2$ for all $A \in \mathscr{A}$. ∎

We remark that if H is simple, U is abelian and $G = U \wr H$, then $G' = G''$. Thus if U is large enough, then G' is a perfect group in which not every element is a commutator.

Finally we mention that one can read off from the character table of a group, the elements which are commutators. In fact $g \in G$ is a commutator iff

$$\sum \chi(g)/\chi(1) > 0,$$

where the sum runs over all complex irreducible characters χ of G.

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RESTRICTIONS ON THE VALUES OF DERIVATIVES

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In **[1]**, F. D. Hammer asked whether there exists a differentiable function f with $f(r)$ rational but $f'(r)$ irrational for every rational r. Posed this way, the problem involves arithmetic properties of the real numbers, and the explicit example constructed by W. Knight **[2]** makes full use of these arithmetic features.

However, the phenomenon under consideration really depends only on the fact that the set of all rational numbers is countable and dense in the line, and that the irrationals are also dense. (Another solution of the problem, found by Dan Simchoni and stated without proof after **[2]**, furnishes an entire function with restrictions of f and f' on an arbitrary countable set, and makes no use of arithmetic.) Once this is recognized, it is easy (as we shall see) to extend this phenomenon to infinitely differentiable functions, in any finite number of variables.

Let n be a fixed positive integer. A *multi-index* is an ordered n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ in which each α_i is a nonnegative integer. To each multi-index α corresponds a differential operator

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

As usual, R is the real line, R^n is euclidean n-space, and $C^\infty(R^n)$ is the class of all functions $f: R^n \to R$ with $D^\alpha f$ continuous for every α.

THEOREM. *Suppose that*

(a) *A is a countable subset of R^n, and*

(b) *for each multi-index α, B_α is a dense subset of R.*

Then there exists an $f \in C^\infty(R^n)$ such that $D^\alpha f$ maps A into B_α, for every α.

Proof. We shall use the customary multi-index notations

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \quad \alpha! = \alpha_1! \cdots \alpha_n!, \quad x^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$$

if $x = (\xi_1, \ldots, \xi_n) \in R^n$.

Arrange the members of A in a sequence $\{x_i\}$, $i = 0, 1, 2, \ldots$, with $x_i \neq x_j$ if $i \neq j$. For $i \geqq 0$, choose $\psi_i \in C^\infty(R^n)$ with compact support K_i, such that

(i) K_i contains no x_m with $m < i$,

(ii) $0 \leqq \psi_i(x) \leqq 1$ for all $x \in R^n$, and

(iii) $\psi_i(x) = 1$ for all x in some neighborhood of x_i.

Choose $c_0(\alpha) \in B_\alpha$, so small that the power series

$$g_0(x) = \sum_\alpha \frac{c_0(\alpha)}{\alpha!}(x - x_0)^\alpha \tag{1}$$

defines an entire function g_0, with $|g_0(x)| < 1$ on K_0. If $f_0 = \psi_0 g_0$ then $f_0 \in C^\infty(R^n)$, and

$$(D^\alpha f_0)(x_0) = (D^\alpha g_0)(x_0) = c_0(\alpha) \tag{2}$$

for every α, since $f_0 = g_0$ in a neighborhood of x_0.