1 Introduction

1.1 A Biography of al-Kāshī and a Brief History

On March 2, 1427 [6, p. 32] and [15], Ghiyāth al-Dīn Jamshīd bin Mas'ūd bin Maḥmood al-Tabīb al-Kāshī (or al-Kāshānī) completed a monumental book in Arabic on arithmetic called *Miftāḥ al-Ḥisāb*. The most commonly used translation of this title, the one which we prefer as well, is "Key to Arithmetic." Other possible translations include "Calculator's Key" and "Key to Calculation." Henceforth we will be referring to it as *Miftāḥ* and its author as al-Kāshī. This encyclopedic work includes three general subjects: arithmetic, geometry, and algebra. It comprises five treatises on arithmetic of integers, arithmetic of fractions, arithmetic of sexagesimal numbers, geometry and measurement, and algebra. Al-Kāshī dedicated the *Miftāḥ* to Ulugh Beg who was the ruler of Samarkand at the time and was a prominent scientist himself. In Berggren's assessment *Miftāḥ* was "the crowning achievement of Islamic arithmetic, and truly a gift fit for a king" [12, p. 22]. Saidan agrees by stating "*Miftāḥ* represents the peak of Islamic arithmetic" [31, p. 29].

Although we do not know all the details of al-Kāshī's life, through his work and letters to his father, one can affirm that he was born in Kāshān, in modern day Iran some 150 miles south of the capital city Tehran. Therefore, he is also known—perhaps more accurately—as al-Kāshānī but al-Kāshī is more commonly found in the literature. He was born in the later part of the fourteenth century. Given al-Kāshī's education and well-documented works at hand, some authors suggest 1380 as a plausible year of his birth (e.g., [37, 38, 24, 16]). According to Suter, al-Kāshī died in the year 1436 [36, 28]. However, based on a note on a manuscript copy of one of al-Kāshī's works, Kennedy proclaims the date of death to be June 22, 1429 (Ramadan 19, 832 AH) [39]. This is the commonly accepted date of al-Kāshī's death.

His given name is Jamshīd, his father's name is Mas'ūd, and his grandfather's name is Maḥmood. He has several nicknames best known of which is al-Kāshī. Giyath al-Din means the rescuer of the religion. Al-Tabīb means physician. This is due to the fact that he practiced medicine as a profession before devoting his time totally to the study of mathematics and astronomy later in his life. He is known as al-Kāshī (or al-Kāshānī) because he is from Kāshān (Fig. 1.1). Assuming a nickname based on the place of birth, tribe of origin, or profession was common in the Arabic and Islamic culture. There are many other famous Islamic scholars who are best known by such nicknames such as al-Khwārizmī and al-Ṭūsī.

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Figure 1.1: Iranian stamp featuring an image of al-Kāshī with an astrolabe in the background

During al-Kāshī's early years, Timur (Tamerlaine, 1336–1405) was conquering and ruling vast lands, including central Iran. Widespread turmoil and poverty afflicted the people of the region [39, 15]. Despite the difficulties at the time, al-Kāshī managed to study science in general, including mathematics and astronomy. In fact, the first dated event that we know of about his life is the observation of a lunar eclipse in his hometown of Kāshān on June 2, 1406, a date recorded in his *Khaqani Zij*[1] as the first date of a series of lunar eclipses he observed. The conditions of the people improved after Shah Rukh Mirza (1377–1447) took over the reign upon his father's death; yet al-Kāshī's best times came only after the young prince Ulugh Beg (1394–1449) became the governor of Samarkand, a city in Transoxiana in modern day Uzbekistan. Ulugh Beg was a recognized scientist himself excelling in mathematics and astronomy who supported many scholars and students. He attracted some of the finest scholars of his time to Samarkand where he first founded a madrasa—a school for advance studies in theology and sciences—between 1417 and 1420, and then establishing the most advanced observatory ever built until that time [20], thus turning Samarkand into a major center of research and learning [6, 5].

We know that al-Kāshī joined the scientific circle of Ulugh Beg upon his invitation around 1420. Scholars reference several dates for this move, 1417 in [37, p. 6], 1418 in [12, p. 21], and 1421 in [9]. Ulugh Beg's invitation of al-Kāshī came after the latter had proven his scientific ability by accomplishing several significant works between 1406 and the date of his move to Samarkand. On June 2, 1406, al-Kāshī observed and recorded the first of a series of lunar eclipses in Kāshān. In 1407 he completed *Sullam Al-Sama*, *Ladder of the Heaven on resolution of difficulties met by predecessors in determination of distances and sizes*. In 1410–1411, he wrote *Mukhtasar dar ilm-i hay'at–Compendium of the science of astronomy–* dedicated to Sultan Iskandar, one of the rulers of Timurid dynasty. In 1413–1414, al-Kāshī wrote *Khaqani Zīj* and dedicated it to Ulugh Beg. In the introduction of this book al-Kāshī complains about his living conditions of poverty while pursuing important work in mathematics and astronomy. He acknowledges Ulugh Beg's support that allowed him to successfully finish this work. Al-Kāshī sought to attract patronage of a ruler, and this might be his first success. In fact, al-Kāshī did some of his best work in Samarkand under the patronage of Ulugh Beg. This includes *Miftāḥ Al-Ḥisab*, an encyclopedic book on elementary mathematics, and his remarkable approximations to π [17] and $\sin(1°)$ [27].

[1] zīj is a book of astronomical tables.

1.1.1 Al-Kāshī's Letters: Invaluable Source

Two letters of al-Kāshī in Persian to his father provide us with interesting facts and insights into the scientific environment of Ulugh Beg's court and personalities of some of the important scholars there. From these letters and other sources we learn that Ulugh Beg tolerated al-Kāshī's lack of court etiquette, thanks to his excellent command of mathematics and astronomy [21, 34, 9].

Al-Kāshī's letters were discovered in the second half of the twentieth century at different times. The second letter was discovered first and published in 1960 independently by two different scholars. Kennedy gave an English translation of the letter together with a commentary in [21], and Sayili gave both a Turkish translation and an English translation in [34]. The first letter was discovered later by Bagheri who published it with an English translation [9].

Al-Kāshī explains that he wrote the second letter and repeated a lot of information in case the first letter, which was sent via the merchants of Qum, might have been lost. A detailed analysis of the letters suggests that al-Kāshī's father was a learned man who knew mathematics and astronomy. The letters' main topic though is al-Kāshī's professional development, accomplishments, and triumphs, assuring his father of his preeminence that clearly distinguishes him from other scientists in the court. Additionally, he speaks highly of the character of his patron Ulugh Beg, and the progress on the observatory that was being built at the time of the writing of the letters (Fig. 1.2).

Figure 1.2: Ulugh Beg's Statue in Samarkand, Uzbekistan (Image source: Wikipedia)

1.1.2 Ulugh Beg in al-Kāshī's Letters

The letters depict Ulugh Beg more as an accomplished scientist and scholar who attended many of the scientific meetings held at his court, than as a ruler or statesman. For example, he mentions that Ulugh Beg is well versed in both religious sciences and mathematics (Translation from [34])

> Truth is that, first of all, he knows most of the holy Quran by heart, and he has a ready knowledge of its exegeses. For each occasion he cites an appropriate verse of the Quran, and he makes elegant quotations. Every day he reads fluently and in the proper manner two sections from the sacred book in the presence of experts who know the whole of the Quran by memory, and no mistakes occur. His knowledge of grammar and syntax is very good, and he writes Arabic extremely well. Likewise, he is well versed in jurisprudence, and he is acquainted with logic and the theory of literary style, as well as with the principles of prosody.
>
> His majesty has great skill in the branches of mathematics. His accomplishment in these matters reached such a degree that one day, while riding, he wished to find out to what day of the solar year a certain date would correspond which was known to be a Monday of the month of Rajab in the year 818 and falling between the 10th and the 15th of the month. On the basis of these data he derived the longitude of the sun to a fraction of 2 minutes by mental calculation while riding on horseback, and when he got down he asked this servant to check his result.
>
> It is true that, as in mental calculation it is necessary to retain quantities in one's mind and to derive others from them, and because in the faculty of memory there is a shortcoming, he [i.e. al-Kāshī himself] could not find the result (correctly) in degrees and minutes and was content with degrees only. But it is not given to any person of our time to do the like; no one else is capable of it.

Ulugh Beg is also described, in these letters, as a kind and open-minded person who is keen on rigorous investigation in science. He listens to all points of views, lets everyone express their opinions, and makes their case, and then engages in arguments with students and experts. He allows discussions to continue until the issues become clear to everyone. He does not approve submission to the authority without convincing proofs [34].

> He is indeed good-natured to the utmost degree of kindness and charity, so that, at times, there goes on, at the madrasa, between His Majesty and the students of the seeker of knowledge so much arguing back and forth on problems pertaining to any of the sciences that it would be difficult to describe it. He has ordered, in fact, that this should be the procedure, and he has allowed that in scientific questions there should be no agreeing until the matter is thoroughly understood and that people should not pretend to understand in order to be pleasing. Occasionally, when someone assented to His Majesty's view out of submission to his authority, His Majesty reprimanded him by saying "you are imputing ignorance on me." He also poses a false question, so that if anyone accepts it out of politeness he will reintroduce the matter and put the man to shame.

Al-Kāshī reveals that Ulugh Beg is very generous and supports a large number of students seeking knowledge (Translation from [9])

> His royal majesty [Ulugh Beg] had donated a charitable gift amounting to 30,000 kopaki dinars, of which 10,000 had been ordered to be given to students. [The names of the recipients] were written down: [thus] 10,000 students steadily engaged in learning and teaching, and qualifying for a financial aid, were listed.

1.1.3 Samarkand in al-Kāshī's Letters: Center of Knowledge

The above-mentioned numbers show that Samarkand was a major center of learning at the time. This is supported by further facts mentioned in al-Kāshī's letters. For example, he states that in addition to the students studying with financial aid, there are about 500 persons among notables and their sons who began studying mathematics at twelve different places. He says "there are 24 calculators some of whom are also astronomers and some began [studying] Euclid ['s *Elements*]" [9]. He compares the situation with Kāshān where one or two individuals may be associated with a given discipline, and concludes that the environment in Kāshān is a fortiori not conducive to healthy scientific discussions.

We also learn that serious scientific studies at Samarkand had been undertaken for about 12 years, and that there were at least 60–70 mathematicians among Ulugh Beg's staff in addition to astronomers.

Not satisfied with zīj's compiled before his time, Ulugh Beg decided to conduct fresh observations and built the Samarkand observatory. Furthermore, we learn that Ulugh Beg visited Maragheh observatory in his childhood which probably made an impression on him. Given what we know, it would be reasonable to infer that the building of the observatory was planned before al-Kāshī's arrival to Samarkand and started shortly thereafter. Astronomy was one of the most important scientific disciplines in the Islamic tradition [32], and observatory was an essential part of that. The significance of observatory in the Islamic world and its legacy in the modern science are summarized by Berggren as "the observatory, as the scientific institution we know today, was born and developed in the Islamic world" [12, p. 21]. The Samarkand observatory that was established by Ulugh Beg turned out to be one of the most important observatories in the Islamic world (Fig. 1.3) as stated by Sayili:

Figure 1.3: Ulugh Beg's observatory in Samarkand, Uzbekistan (Image from Wikimedia commons)

> From the standpoint of longevity and work therefore the Samarqand Observatory was one of the most important observatories of Islam, and it probably was the most important... Giyāth al-Din Jamshīd and Muin Al-Din-i Kāshī prepared the plan for the observatory... Giyath Al-Din was the first director of the Samarqand observatory ... [33, p. 265, 266, 271].

We learn from al-Kāshī's letters to his father that Ulugh Beg followed al-Kāshī's recommendations on certain technical aspects of the building upon being convinced by al-Kāshī's explanations. This led to changes to the original plan during the construction. Al-Kāshī's letters are an invaluable firsthand source of information on the instruments used at the Samarkand observatory.

As stated earlier, a major point al-Kāshī makes in his letters is to reassure his father on al-Kāshī's own elevated status, and prominence among his peers. He conveys that he is the most proficient and competent scientist in the court. Reading his letters, he might be perceived as boastful, yet we believe it is rather a sign of competence mixed with confidence.

Al-Kāshī narrates how on several occasions other scientists could not solve a given problem even after spending a considerable time on it, yet when he tackles that given problem, he is able to solve it easily. In other instances, when some scholars think that there is a missing information in the problem statement, al-Kāshī shows that this is not the case and solves the problem. Other times, he publicly challenges other scholars on scientific issues and proves his point. He claims this made him famous, and to support his claim he writes of strong words of praise for himself by Ulugh Beg about his character and scientific competency [34].

> As to the complimentary remarks of His Majesty, of which mention has been made above, the situation is that no week passes without some friends reporting to this servant that His Majesty made such and such remarks tonight or today concerning me. They are as follows: "He has the knowledge of things ready at hand"; "He knows extremely well"; "His knowledge is superior to that of others"; "His knowledge is more readily at his disposal and more substantial than is the case with Qāḍīzāde"; 'His mind works better in this science than that of Qāḍīzāde "; "Mawlana Giyāth al-Dīn knows all the parts of this science, and he solves at once or in a single day a difficulty which takes Qāḍīzāde ten days to disentangle."
>
> He said, likewise, "He is a good and kind-hearted man. All those who have access to our circle, whether they be of the notables or not, have not restrained themselves but have quarreled with people and have transgressed their limits although I have shown them little courtesy. On the other hand, although I have extended much courtesy and many grants to Mawlana Giyath Al-Din and even though he is always honored with access to our company, he has not had any quarrels with anybody and nor has anyone complained about him." "He has not, out of greed, resorted to speak and gossip behind people's backs"; "He conducts himself very well." He has said things of this nature many times.

On the other hand, in describing the majority of scientists in the court, al-Kāshī often refers to their incompetence in disdain [15]. This is certainly a point that he uses to set himself, his patron Ulugh Beg along with Qāḍīzāde al-Rūmī (1364–1436), and very few other scientists apart. Al-Kāshī holds Qāḍīzāde in high esteem and considers him as the most knowledgeable scholar in the group. At the same time, al-Kāshī narrates situations where he excelled better than Qāḍīzāde. An obvious logical consequence of this is that al-Kāshī is the very best scholar in the entire group of Ulugh Beg's scientific staff. He informs his father that he followed his advice to focus on only one subject (in this case it is astronomy) and he understands the reason behind this advice. He stated: "because occupation with another subject would indeed distract me from astronomical observations; second, because of my occupation with another art in which I may be a beginner, there may occur in my discussions or compositions some defect or error which people would bring to bear on the other arts [in which I am adept]" [9]. Evidently, another purpose of al-Kāshī's letters to his father was to counter and refute some rumors his father heard through a person named Shams al-Din. He makes it clear that Shams al-Din's statements about him are in fact false.

Al-Kāshī had a productive time in Samarkand and was the first director of the astronomical observations at the observatory [34]. Unfortunately, he passed away on June 22, 1429 at the observatory in which he was deeply involved, before the full study came to fruition. Qāḍīzāde al-Rūmī took over but sadly, his life was not long enough either to complete the project. Finally, the project was completed under the direction of Ali Qushji (1403–1474).

In the preface of his own zīj written some years after al-Kāshī's death, Ulugh Beg wrote the following testimony for al-Kāshī: "the remarkable scientist, one of the most famous in the world, who had a perfect command of the sciences of the ancients, who contributed to its development, and who could solve the most difficult problems" [39].

In the twentieth century, many scholars recognized al-Kāshī's work in general and *Miftāḥ* in particular. For example, Fuat Sezgin, one of the prominent researchers in the history of Islamic science, summarized it all as:

> The achievements of al-Kāshī represent the culmination of Islamic civilization's progress in mathematics. Some of these achievements reappeared or rediscovered later in Europe. [35, p. 66]

1.2 List of al-Kāshī's Known Works

In addition to *Miftāḥ* which is the most comprehensive work of al-Kāshī, we know of the following works of al-Kāshī. They were collected in *Majmu'*(*Collection*), Tehran 1888. The following list of al-Kāshī's works can be found in [6, 39].

- **Zij Khaqani**

  Zīj' is a Persian word used for astronomy books containing tables that are important for astronomical calculations. This work was written around 1413–1414 before al-Kāshī moved to Samarkand. Written in Persian, it was an update and completion of the *Zij Ilkhani* compiled by famous Nasir al-Din Al-Tusi about 150 years earlier. Bartold believed that al-Kāshī dedicated this work to Shah Rukh who was a patron of science in Herat [10]. However, Kennedy established that the zij was dedicated to Ulugh Beg, son of Shah Rukh and ruler of Samarkand [39]. This may be al-Kāshī's first attempt to secure funding under Ulugh Beg's patronage. This zij contains six treatises and the tables in it are in sexagesimal system [39]. Al-Kāshī used iterative methods to obtain approximate value of the third of each arc [6]. He also states that he collected data from the works of earlier astronomers (astrologers) that did not show up in other tables, along with geometric proofs [6]. Arabic manuscripts are available in London, Oxford, and Istanbul [39].

- **The Treatise on Circumference**

  One of al-Kāshī's most remarkable achievements is his approximation of π accurate to 9 sexagesimal places, or 16 decimal places. According to Hogendijk [17], this is one of the highlights of the medieval Islamic mathematical tradition. Before al-Kāshī, the previous record was 6 decimal place approximation by Chinese mathematicians. It took nearly two centuries before the Dutch mathematician Ludolf Van Ceulen surpassed al-Kāshī's accuracy with 20 decimal places. Written in 1424, the original title of this work is *Al-risala al-muhitiyya*. His motivation to compute π so accurately was to calculate the circumference of the universe so that the round-off error in the approximation would be no more than the width of a horse's hair. He set the accuracy of his approximation in advance and he used a polygon with $3 \times 2^{28} = 805,306,368$ sides. Hogendijk states that since al-Kāshī's text was not available in English translation until recently, incorrect or confused statements on the history of π often appeared in the western literature [17]. For example, al-Kāshī was not mentioned in *A History of Pi* [17], a popular book on the subject. Arabic manuscripts are available in Istanbul, Tehran, and Meshed [39].

- **The Treatise on Chord and Sine**

  Given al-Kāshī's dissatisfaction with earlier zijs, he calculated accurate values for sines and chords in this work to help with astronomical tables. He also calculated $\sin(1°)$ to the same degree of accuracy of his computation of π. Unfortunately, the original text of this work is lost; however, Qāḍīzāde al-Rūmī's account of this treatise is available in the national library of Iran [8] and is translated to Russian as well. The outline of al-Kāshī's main method in this approximation is as follows. He first used the trig identity $\sin(3\theta) = 3\sin(\theta) - 4\sin^3(\theta)$ which he transformed to a cubic equation of the form $x = a + bx^3$. He then used an iterative method to obtain successive approximations to $\sin(1°)$ that get more accurate at each step. His beautiful approach is still important in modern mathematics and is usually covered under the name of "fixed point iteration" in numerical analysis courses. See [1] for more details on the computation of $\sin(1°)$. There is an edition of this manuscript in *Majmu'* [39].

- **The Zīj at-Tashilat (The Zīj of Simplifications)**

  Al-Kāshī refers to this work in *Miftāḥ* among his works but it is nonexistent. It probably included a simplified method of computing the positions of celestial bodies [20].

- **Risala dar Sharh-i Alat-i Rasad (Treatise on the Explanation of Observational Instruments)**

  Written for Sultan Iskandar (Kara Yusuf of Black Sheep Turks (Karakoyunlu) dynasty) in 1416, al-Kāshī gives brief yet accurate descriptions of the constructions of eight astronomical instruments. A manuscript copy of this work is available in Leiden, which, according to F. F. Bartold, was written by al-Kāshī himself [6]. Moreover, Persian manuscripts are available in Leiden and Tehran[39].

- **Nuzha Al-Hadaiq (Delight of Gardens)**

  Al-Kāshī wrote this treatise in 1416 (February 10, 1416) in Kāshān to which he made additions in Samarkand in 1426. A revised Persian version was written by an anonymous astronomer in Istanbul around 1490 [18]. It gives an explanation of how to build an instrument invented by al-Kāshī which he called "The Plate of Conjunctions" [6]. According to al-Kāshī, it is an instrument from which one can retrieve the calendars of planets, their widths, dimensions, distances from earth, eclipses, and other related matters. One can describe this instrument, which has a similar shape as the astrolabe, as the diagram for the approximate graphical solutions of many problems related to the movement of the stars based on the mean values of their coordinates. Al-Kāshī's later additions include ten appendices which describe additional techniques to utilize the instrument [20]. Arabic manuscripts are available in London, Dublin, and Bombay [39].

- **Sullam Al-Sama (The Ladder of the Sky or The Stairway of Heaven)**

  al-Kāshī completed this work in his hometown in 1407. It is the earliest known work of al-Kāshī.The full title of this text in the area of astronomy is "on Resolution of Difficulties Met by Predecessors in Determination of Distances and Sizes (of Heavenly Bodies)" [24]. Al-Kāshī mentions this treatise in the preface of *Miftāḥ*. He implies that scholars before him had difficulties and disagreements about distances and sizes of the heavenly bodies. So, he decided to write this book in order to help future scholars. Arabic manuscripts are available in London, Oxford, and Istanbul [39]. There is an Iranian TV series with the same title, the Ladder of the Sky, about al-Kāshī's life which was broadcast during the month of Ramadan of 2009. Full episodes are available at http://www.shiasource.com/ladder-of-the-sky

- **Talkith Al-Miftāḥ (Abridged Key [to Arithmetic])**

  Written before the *Miftāḥ* itself (in 1421), it was an early and abbreviated version which contains three treatises and about one-eighth of the material in *Miftāḥ* [8]. Qurbani provided Persian translations of the chapters of *Talkhith* [25]. Arabic manuscripts are available in London, Tashkent, Istanbul, Baghdad, Mosul, Tehran, Tabriz, and Patna [39].

- **Miftāḥ al Asbab-fi al-Ilm al-Zīj (The Key of Reason in the Science of Astronomical Tables)**

  There is an Arabic manuscript in Mosul (120/306) [39, 28].

- **Risala dar Sakht-i Asturlab (The Treatise on the Construction of an Astrolabe)**

  There is a Persian manuscript in Mosul [39].

- **Ta'rib Al-Zīj (Arabization of the Zīj)**

  This is an Arabic translation of the introduction of Ulugh Beg's zīj which was in Persian. Manuscripts are available in Leiden and Tashkent [39].

- **Ilḥāqāt an-Nuzha (Supplement to the Excursion, 1427)**

  There is an edition of a manuscript in *Majmu'* [39].

- **Wujūh al-Amal al-Darb fī'l-Takht wa'l-Turāb (Ways of Multiplying Using a Dust Board)**

  There is an edition of an Arabic manuscript in *Majmu'* [39].

- **Natāij al-Ḥaqāiq (Consequences of Truths)**

  There is an edition of an Arabic manuscript in *Majmu'* [39].

- **Risāla fī Ma'rifa Samt al-Qibla min Dāira Hindiyya Ma'rufa (Treatise on the Determination of Direction of the Qibla from a Circle Known as Indian Circle)**

  There is an edition of an Arabic manuscript in Meshed (84) [39] [28].

- **Tuḥfat al Sultān fī Asbab al-Irfān (Gift to Sultan on Causes of the Science)**

  Oxford (1514). Treatise on astronomy dedicated to Amir-zade Ibrāhīm Sultan, son of Shahruh [28].

- **Risala dar Hay'at (Treatise on Astronomy)**

  P-London (Sup. 27261), Yazd (Jami 439/5) [28].

- **Mukhtasar dar Ilm-i Hay'at (Concise Treatise on the Science of Astronomy)**

  P-London (869b), Treatise dedicated to Amir-zade Sultan Iskandar Bahadur [28].

- **Risālat ḥall ashkāl Utārid (Treatise on Solution of Propositions on Mercury)**

  Meshed (5527) [28].

- **Risala-yi Kamaliyya (Treatise for Kamal al-Din)**

  P-Hyderabad (riyada 125–126), Astronomical treatise in seven books and a conclusion written in Kashan in 1406, dedicated to vizier Kamal al-Din Mahmud [28].

- **Risāla fī Tashīh Awsat al-Qamar min al-Arsad al-Khusufiyya (Treatise on Closer Definition of the Center of the Moon from Observations of Eclipses)**

  Cairo (riyada 898/23) [28].

- **Zīj al-Kāshī**-Meshed (5321) [28]

- **Notes on Linear Interpolation**—Cairo (Zaki 917/14) [28]

1.3 Manuscript Copies of *Miftāḥ*

Two different typeset and printed books of *Miftāḥ* that include the original Arabic manuscript together with some commentary and explanations are available. The first one is by Ahmad Sa'id Al-Dimirdash and Muhammad Hamdi Al-Hifni al-Shaykh, published in Cairo in 1969 [5] in which commentaries are in Arabic. The other one was published in 1977 in Damascus by Nabulsi [6]. Nabulsi edition has most commentaries in Arabic and some in French as well. It lists the following known manuscript copies of *Miftāḥ* at the time of its publication.

1. Original book of Miftāḥ Al-Ḥisab, written by Jamshīd bin Mas'ud bin Mahmoud al-Kāshī March 3rd, 1427 (830 h.), but it is missing.

2. Al Burgandy's manuscript, which is also missing. According to Muhammad Al-Sadiq Al-Arassengi in the Zahiri manuscript, it is written by Abdul Ali Al-Burgandy on Tuesday, 17th day of Dhu Al-Hijja in 889 H., which corresponds to January 5, 1485.

3. The Leiden manuscript, written in 1558 (965 h).

4. The British Museum manuscript, written in 1589 (887 h).

5. The Zahiri manuscript written in 1691 (1102 h).

6. The Leningrad manuscript, written in 1789 (1204 h).

7. The Scientific Library of Prussia manuscript in Berlin, written in 1886 (1303–1304 h).
8. The Public Scientific Library in Berlin manuscript (spr 1824)
9. The Berlin Institute for History of Medicine and Science, number 1 and 2.
10. The National Library of Paris, number 5020.

Dimirdash and Muhammad Hamdi edition also mentions the manuscripts 3,4,6–10. They state that they relied on Leiden manuscript [5]. Their list does not include the Zahiri manuscript but includes one that does not appear in the list above, namely, a stone-print copy in Tehran, located in the Tayomrian cabinet number 255-Math. Nabulsi says he used the following manuscripts: Zahiri, London, and Leiden as well as the print book [5] and the Russian translation by Rosenfeld and Youschkevitch published 1956 [29]. The Zahiri manuscript is the main source of his investigation. Nabulsi regards Zahiri manuscript as the oldest known one, gives the following information about it [6, p. 31], and includes a picture of the first and last pages.

> The Zahiri manuscript, number 7795 amongst the collection of books in the Zahiri library in Damascus. It is written on 128 sheets of paper of dimensions $21 \cdot 12$ cm, the handwriting is nice, the titles are written in red, has commentaries, it has eroded from the sides, embroidered with gold (written by Abd Al Ali bin Muhammad Al Burgandi in 889 then copied by Ibn Muhammad Jafar Sadiq Al Arasinji in 1103).

Nabulsi notices that time intervals between the original work and its subsequent manuscripts listed above are 73 years, 31 years, 102 years, 98 years, and 97 years. He concludes that there must be several other manuscripts of *Miftāḥ* that are still missing to this day. The manuscript we are using is not in Nabulsi's list, and it appears to be one of the manuscripts he suspected was missing. This manuscript is in Süleymaniye library in Istanbul in its Nuruosmaniye branch with new record number 2532 and old record number 2967. It was written in Ramadan 854 H., which corresponds to October 1450. There are several other manuscripts of *Miftāḥ* in Süleymaniye library which are labeled as Atıfefendi1719, Esadefendi13175, Fatih5421, Hamidiye883, and Hüsnüpaşa1268. Some of them appear to be incomplete and some of them do not seem to contain a date of completion. The manuscript that we are using, Nuruosmaniye2967, is the oldest one in the collection, and it is complete. Based on available data in the literature, Nuruosmaniye2967 manuscript seems to be one of the oldest, possibly the oldest, available manuscript copy of *Miftāḥ* to this date. Note that the Zahiri manuscript that Nabulsi considers "the oldest known manuscript of *Miftāḥ* so far" was originally written in 889 H. (1484 CE) which is later than the date of Nuruosmaniye manuscript we use (854 H-1450 CE). The only reference in the literature to the Nuruosmaniye manuscript that we are aware of is in [31]. The translators of section 5 of treatise 3 of Miftāḥ (on root extraction in sexagesimal system) used the following three manuscript copies.

- Princeton University MS ELS 1189
- India Office Arabic MS L756
- University of Leiden MS Or. 185.

1.4 Pedagogical Aspects of *Miftāḥ*

It is clear that *Miftāḥ* was written as a textbook and a practical guide for people in various professions that need arithmetic and elementary mathematics as mentioned by al-Kāshī himself in the introduction:

> I solved many problems I was asked by expert mathematicians either for testing me or for their own learning. Some of those problems could not be solved by one of the six algebraic forms. Through these works I acquired a lot of knowledge to solve elementary mathematical problems in the easiest, most beneficial and most efficient ways using clear methods. I wanted to clarify and compile them so that it becomes a guide to anyone interested. Therefore, I wrote this book and collected all that professional calculators need, avoiding tiring length and annoying brevity. I presented rules for most operations in tables to make them accessible for engineers.

It is also clear that *Miftāḥ* is not a theoretical book as the author did not make any effort to prove or justify the algorithms or procedures even by the standards of that time. The title suggests that arithmetic is viewed as the key in solving any problem that requires calculations. In the introduction of *Miftāḥ*, al-Kāshī describes arithmetic as follows: "Arithmetic is the science of rules to determine numerical unknowns from specific known quantities." Youschkevitch and Rosenfeld summarized the essence of the book by stating:

> In the richness of its contents and in the application of arithmetical and algebraic methods to the solution of various problems, including several geometric ones, and in the clarity and elegance of exposition, this voluminous textbook [that is, *Miftāḥ*] is one of the best in the whole of medieval literature; it attests to both the author's erudition and his pedagogical ability. Because of its high quality the Miftāḥ was often recopied and served as a manual for hundreds of years. [39, p. 256]

Taani picked on the *pedagogical ability* and further studied the content and pedagogy of *Miftāḥ* [37, 38]. His main findings about al-Kāshī's pedagogy in *Miftāḥ* are what he calls "multiple paths to practice mathematics" and his "exhaustive exclusive classification methods." Taani's formulation of "multiple paths" include the following features: multiple definitions, multiple algorithms, multiple formulas, and multiple solutions [38]. He explains that what inspired him to consider investigating multiple paths in al-Kāshī's pedagogy was student comments. He initially used some excerpts from al-Kāshī in his precalculus class and his goal was to "get data from students about using historical sources in the classroom." When he administered a questionnaire about the lesson, student comments on al-Kāshī's method of using multiple methods/approaches to solve a given problem captured his attention and that became a major topic in his dissertation.

As an example of multiple definitions of mathematical concepts, consider these two definitions of division by al-Kāshī in *Miftāḥ*

1. In integers, it is partitioning of the dividend in units of the divisor into a number of equal parts so that each share from the divisor is fixed. Such a share is called the quotient.

2. Its general definition is to obtain the number whose ratio to one is the same as the ratio of the dividend to the divisor.

Al-Kāshī often presents multiple methods to perform mathematical calculations. For example, he presents five different ways of performing multiplication of integers. He gives three different formulas to compute the area of a generic triangle, and four different formulas specifically for computing the area of an equilateral triangle. The last treatise of *Miftāḥ* contains a number of word problems and problems in inheritance (determining the share of each heir according to Islamic inheritance laws). He shows multiple ways of solving these problems.

Taani suggests that his work may be used in a classroom in two different ways [38]. The first is directly using the primary text of al-Kāshī will let the students live the experience of discovering mathematics in the fifteenth century. The second is to follow al-Kāshī in presenting certain topics from multiple perspectives. He comments that the benefits of using multiple paths include: *providing flexibility of thinking*, *providing opportunities for comparison*, *creating a network of ideas*, and *improving creativity* [38].

Therefore, in addition to its profound significance in the history of mathematics due to its content, *Miftāḥ Al-Ḥisab* is also worthy of attention for its pedagogical aspects.

1.5 Possible Future Projects

As mentioned earlier, not only does al-Kāshī not give proofs or justifications for his algorithms and procedures, but he does not give any information about their origins either. It is not clear what parts are his inventions or contributions and what parts are considered established results. A possible future work could be to research the history of the algorithms (such as finding square roots and higher degree roots) described in *Miftāḥ* to determine what exactly the contributions of al-Kāshī might be. On a different direction, it is quite likely that al-Kāshī's work may have influenced mathematicians who came after him. In a recent article [7] we pointed out one such possible connection. We observed that the famous Flemish mathematician Simon Stevin (1548–1620) presented root finding algorithms in his well-known book *L'arithmétique"* ("Arithmetic"), published in 1585. On the surface, Stevin's algorithms look much different from al-Kāshī's. However, after careful examination, we observe that the underlying algorithm is the same, though there are some curious differences in details and implementation (see [7] for more info). To the best of our knowledge, no direct connection between the two mathematicians is known. Given al-Kāshī's influence on Ottoman mathematicians and the education system [19], one might imagine flow of ideas from Central Asia to Europe through the Ottoman land. This is definitely an area that requires further research.

1.6 Notes on Translation and the Purpose of This Work

This work is a translation with commentaries of the book *Miftāḥ Al-Ḥisab* written by al-Kāshī on March 2, 1427. We are using what we believe to be the oldest manuscript so far at hand, namely the Nuruosmaniye2967 manuscript which was originally written in 854 H.-1450 CE [2]. Believing that this book would be a reference to researchers and students of history of Islamic Mathematics alike, we tried to strike a balance between a literal translation and a smooth translation in modern English for the convenience of the reader. On one hand, we strove to preserve the authenticity of the manuscript with a faithful translation that conveys all possible meanings that al-Kāshī might have had in mind, and on the other hand, we tried to provide a meaningful text to a modern reader. So, we resorted to footnotes and square brackets in the text to further explain some of the ambiguities and odd to read sentences. Any inserted text from the margins of the manuscript are included in square brackets with footnote assertion to that effect. Also, we underlined some terms to indicate they are further explained in the glossary. Since Arabic is a language written and read from right to left, some tables have been reverted to read from left to right, except when there are no ambiguities the tables are left as they are. For example, in tables about addition or multiplication, numbers are written and operated in Arabic the same way as in English. Hence, they are left as they are. However, tables and numbers with sexagesimal digits are reverted when degrees, minutes, and seconds are not explicitly mentioned since they are in reverse order in Arabic and English. When the degrees, minutes, and seconds are explicitly written with no confusion we leave the numbers and tables as they are. Moreover, many of the numbers and tables in the manuscript have been checked against several other manuscripts for accuracy. We indicate that in footnotes. Still, we recommend caution when dealing with complex computations as some differences and ambiguities might rise between different manuscripts or even in different places within the same manuscript as the handwritten letters and numbers might lead to some confusions.

Our goal in this work is to present the raw material of *Miftāḥ Al-Ḥisab* with some directions that could possibly enrich further studies by students and researchers. We purposely refrained from making any judgments on the nature, origin, or the originality of the subject matter found in the *Miftāḥ*. Some of al-Kāshī's work was in fact original, others were known to previous authors centuries ago. We leave the critical in-depth historical study to the invested reader to interpret, position, and effectively study al-Kāshī's work in its own right and/or in a historical context.