Ibn al-Haytham's Lemmas for Solving "Alhazen's Problem"

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I

"Alhazen's problem",* or "*problema Alhaseni* (or *Alhazeni*)", is the name given by seventeenth-century mathematicians to a problem which they encountered in the *Optics* of AL-ḤASAN IBN AL-HAYTHAM. The *Optics*, composed in the first half of the eleventh century, had been translated into Latin in the late twelfth or early thirteenth century,[1] and an edition of it by FRIEDRICH RISNER had been published at Basel in 1572.[2] CHRISTIAAN HUYGENS formulated the problem as

* A shorter version of this paper was read at the annual meeting of the History of Science Society which took place in New York in December 1979. I am grateful to A. ANBOUBA, J. L. BERGGREN, J. P. HOGENDIJK and E. S. KENNEDY for comments, suggestions and corrections on all or part of this paper. All errors and shortcomings that remain are of course my own. The attached translation of IBN AL-HAYTHAM's lemmas is part of a project involving an edition and English translation of the Arabic text of IBN AL-HAYTHAM's *Optics* (*Kitāb al-Manāẓir*). I wish to thank the U. S. National Science Foundation and the National Endowment for the Humanities, for their support of this research.

[1] Neither the name of the translator(s), nor the place or exact date of the translation has been ascertained. Of the twenty odd manuscripts that have been located in European libraries, the earliest are from the thirteenth century, and one of these (the Edinburgh Royal Observatory MS CR3.3 = MS 9–11–3(20)) is dated 1269 (see D. C. LINDBERG, *A Catalogue of Medieval and Renaissance Optical Manuscripts*, Toronto: The Pontifical Institute of Medieval Studies, 1975, pp. 17–19). The earliest mention of the Latin version of the *Optics* in the West occurs in a work by JORDANUS DE NEMORE who flourished in the period between 1220 and 1230 (see MARSHALL CLAGETT, *Archimedes in the Middle Ages*. Vol. I: The Arabo-Latin Tradition, Madison: University of Wisconsin Press, 1964, pp. 668–9 and 674).

[2] *Opticae thesaurus. Alhazeni Arabis libri septem, nunc primum editi. Eiusdem liber De crepusculis et nubium ascensionibus. Item Vitellonis Thuringopoloni libri X. Omni instaurati, figuris illustrati et aucti, adiectis etiam in Alhazenum commentariis, a Federico Risnero*. Basel, 1572. (Reprinted, New York: Johnson Reprint Corporation, with a valuable Introduction by D. C. LINDBERG.) '*Opticae thesaurus*' is clearly the collective title of the whole volume and should not be cited as the title of ALHAZEN's 'seven books,'

that of finding the point of reflection on the surface of a spherical mirror, convex or concave, given the two points related to one another as eye and visible object.[3] He had found IBN AL-HAYTHAM'S treatment of the problem "too long and wearisome" (*longa admodum ac tediosa*),[4] and, armed with the tools of modern algebra and analytic geometry, he set out to produce a solution of his own—a task which he finally fulfilled to his own satisfaction in 1672, having proposed an earlier solution in 1669.

"Long and wearisome" though IBN AL-HAYTHAM'S treatment may have been, it certainly represented one of the high achievements of Arabic geometry, and its importance for the history of mathematics in Europe down to the seventeenth century is easily recognizable. HUYGENS' brief and elegant solution was itself based on the same idea which IBN AL-HAYTHAM had used six hundred years earlier—the intersection of a circle and a hyperbola.

This paper is concerned with "Alhazen's problem" as it appears in IBN AL-HAYTHAM'S *Optics*. The problem of finding the reflection-point occurs in this book as part of a long series of investigations of specular images which occupy the whole of Book V, and these investigations in turn presuppose a theory of optical reflection which is expounded in Book IV. Much of the character of IBN

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as is often done. The seven books were together known in the Middle Ages as *Perspectiva* or *De aspectibus*, the titles sometimes shown in the extant manuscripts. It may be interesting to note that when the emir (or admiral) EUGENE OF SICILY translated PTOLEMY'S *Optics* from the Arabic into Latin in the twelfth century, he chose as the title the original Greek '*Optica*' rather than any Latin rendering of the Arabic '*al-manāẓir*' (see *L'Optique de Claude Ptolémée dans la version latine d'après l'arabe de l'émir Eugène de Sicile*, edition critique et exégétique par ALBERT LEJEUNE, Louvain: Bibliothèque de l'Université, 1956). EUGENE, whose native tongue was Greek, had access to the Greek text of EUCLID'S *Optica* which, like the works of PTOLEMY and IBN AL-HAYTHAM, was called in Arabic *Kitāb al-Manāẓir*. On EUGENE see C. H. HASKINS, *Studies in the History of Medieval Science*, New York: Frederick Ungar Publishing Co., 2nd ed., republished 1960, pp. 171 ff.

[3] See *Oeuvres complètes de Christiaan Huygens*, vol. XX (Musique et Mathématique Musique. Mathématiques de 1666 à 1695), La Haye, 1940, pp. 207, 265–71, 272–81, 328, 329, and 330–33; see especially p. 265. In 1669 HUYGENS expressed the problem in optical terms: "Dato speculo sphaerico convexo aut cavo, datisque puncto visus et puncto rei visae, invenire in superficie speculi punctum reflexionis" (*ibid.*, p. 265). In 1672 the formulation became purely mathematical: "Dato circulo cujus centrum *A* radius *AD*, et punctis duobus *B*, *C*. Invenio punctum *H* in circumferentia circuli dati, unde ductae *HB*, *HC* faciant ad circumferentiam angulos aequales" (*ibid.*, p. 328; also vol. VII, pp. 187–9). See note 4 below.

[4] *Ibid.*, p. 330. ISAAC BARROW was another mathematician in the seventeenth century who was annoyed by the excessive length of IBN AL-HAYTHAM'S solution. In Lecture IX of his *Lectiones XVIII cantabrigiae in scholis publicis habitae* (first published at London in 1669), he described IBN AL-HAYTHAM'S demonstrations as "horribly prolix" (see p. 74). Neither HUYGENS nor BARROW was, however, concerned to explain the character (objectionable or otherwise) of IBN AL-HAYTHAM'S method of solution. Their approach was that of mathematicians, not of historians of mathematics. See the relevant remarks by SABETAI UNGURU in his edition and English translation of *Witelonis Perspectivae liber Primus* (Studia Copernicana XV), Wrocław, *etc.*: Ossolineum (The Polish Academy of Sciences Press), 1977, pp. 209–12.

AL-HAYTHAM'S treatment of reflection-points can only be appreciated if understood with reference to this wider context. It should also be mentioned that IBN AL-HAYTHAM'S researches extended to cylindrical and conical as well as spherical mirrors. IBN AL-HAYTHAM was therefore aiming to solve a wider and more complex set of problems than "Alhazen's problem" in HUYGENS' limited sense. Here, however, I am only concerned to give an account of that aspect of IBN AL-HAYTHAM'S treatment which can be directly related to HUYGENS' formulation, and to present a full translation of the six lemmas which IBN AL-HAYTHAM proposed for solving the problem in all its generality. The clarifications which I hope to make are intended to be part of a more comprehensive study.

The limited problem with which we shall be concerned is, therefore, that of finding the point of reflection on the surface of a spherical mirror. Let us begin with IBN AL-HAYTHAM'S solution as applied to the case of a convex mirror.

A and B (in Fig. 1.1) are, respectively, the given locations of the eye and the visible point. G is the centre of the mirror with a radius GD, given in magnitude. The plane of the circle is that containing lines AG, BG; and it is proposed to find on the circumference of the circle a point D, such that AD and DB will make equal angles with the tangent at D.

IBN AL-HAYTHAM takes at random a line MN (Fig. 1.2), which he divides in a point F, such that

$$\frac{MF}{FN} = \frac{BG}{GA}.$$

From point O at the middle of MN he draws the perpendicular OC, on which he takes a point C, such that

$$\sphericalangle OCN = \frac{1}{2} AGB.$$

Then, and this is the crucial step, through F, he draws line QFS, cutting NC in Q and the extension of CO in S, so that

$$\frac{SQ}{QN} = \frac{BG}{GD}.$$

Now IBN AL-HAYTHAM shows, *before coming to this proposition*, that two such lines can be drawn through F, producing two unequal angles at N. He takes the case of the larger of the two angles and further assumes that angle SNQ is obtuse. (I have reversed the order of presentation to spare the reader some of the suspense, but I shall return to this crucial construction.)

Having made this assumption, the construction of Figure 1.1 proceeds as follows:

Draw GD at an angle BGD equal to SQN: this gives the position of D which is now to be shown to be the point of reflection of the light from B to A.

IBN AL-HAYTHAM continues as follows: He produces GD to E and draws line ZDT tangent to the circle at D.

He then draws DK at an angle GDK equal to angle QNF (Fig. 1.1), and BR perpendicular to the extension of DK. (He can do the latter because angle GKD is acute.)

He further extends DR to I, so that IR is equal to RD, and joins BI.

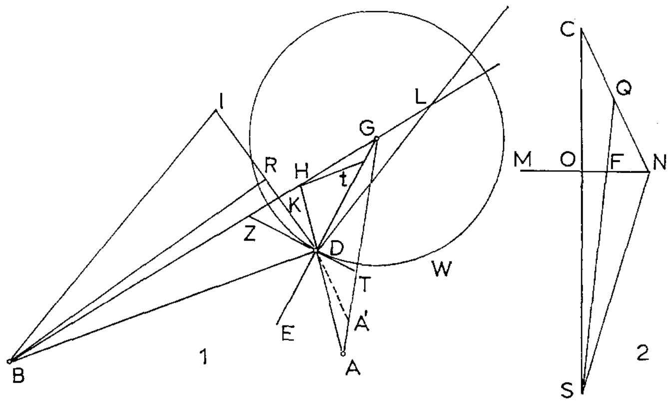

Fig. 1

Finally, he draws *DL* parallel to *BI*, constructs angle *LDH* equal to *AGB*, and draws *Ht* parallel to *BD*.

He proves that the extension of *HD* cuts *GA* at precisely point *A*, and finally deduces the equality of the angles made by *AD* and *BD* with *DE*, the normal to the tangent at *D*.

Figure 1 is not shown in the extant manuscripts of Book V of IBN AL-HAYTHAM'S *Optics*. It is here constructed from the edited text of the *Optics*.[5] The inferred figure is essentially similar to the corresponding figures in KAMĀL AL-DĪN'S commentary[6] and RISNER'S edition of the medieval Latin translation, but is not identical with them.

I have deliberately added only one feature—the discontinuous line *DA'* as a *hypothetical* rectilinear extension of line *HD*. This merely simplifies the language of the proof without altering it in any other way.

Let us, then, say that *HD* produced cuts *GA* at point *A'*.

To prove that *A'* coincides with *A*, and, therefore, that *HDA* is a straight line, IBN AL-HAYTHAM has to show that *GA'* is equal to *GA*.

This he does by first considering triangles *DHL* and *GHA'*, which are similar *by construction*, and this gives him:

$$\frac{DH}{DL} = \frac{HG}{GA'}.$$

Then he shows, again by consideration of similar triangles, that

$$\frac{DH}{DL} = \frac{HG}{GA}.$$

From which it follows that

$$GA' = GA.$$

[5] See below, n. 16.

[6] The "Commentary" by KAMĀL AL-DĪN AL-FĀRISĪ on IBN AL-HAYTHAM'S *Kitāb al-Manāzir*, known as *Tanqīḥ al-Manāzir*, is believed to have been completed around A.D. 1300. KAMĀL AL-DĪN died in A.D. 1320. The *Tanqīḥ* has been published in an unsatisfactory edition in two volumes at Hyderabad, Dn. in 1928–1930.

(His proof involves taking HD as a mean proportional between BD and DL, *i.e.*

$$\frac{BD}{DL} = \frac{BD}{HD} \cdot \frac{HD}{DL} \left(= \frac{BG}{GA}\right), \tag{1}$$

and HG as a mean proportional between BG and GA, *i.e.*

$$\frac{BG}{GA} = \frac{BG}{HG} \cdot \frac{HG}{GA}. \tag{2}$$

Since

$$\frac{BD}{HD\,(=Ht)} = \frac{BG}{HG},$$

it follows, by substitution in (1), that

$$\frac{DH}{DL} = \frac{HG}{GA}.) \tag{3}$$

Let us now return to the construction of the key figure on the right. IBN AL-HAYTHAM'S chief contribution to the solution of this problem (and of the larger problem of finding the reflection-point or points on the surface of mirrors of other shapes) consists in the formulation and proof of six propositions or, as he properly calls them, lemmas (*muqaddamāt*)[7] which form the basis of his proofs. Except for elementary cases, some of which had been treated by PTOLEMY,[8] all constructions of reflection-points are presented by him as applications of these lemmas. In modern accounts of IBN AL-HAYTHAM'S theory of optical reflection these lemmas are either ignored, cursorily dealt with, or re-formulated in modern terms. In what follows I shall try to keep as close as possible to IBN AL-HAYTHAM'S procedure, my aim being largely to guide the reader through IBN AL-HAYTHAM'S text.

Figure 2 is not from the *Optics*; it is a modern representation of two of IBN AL-HAYTHAM'S lemmas, the first and the second. I have chosen to start with this figure because, being modern, it is quickly understandable, and it has the advantage (from the point of view of historical analysis) of being close to IBN AL-HAYTHAM'S own figures. It is here reproduced, with some changes, from the important study published by M. NAẒĪF in 1943.[9]

[7] I write *muqaddamāt* (in the passive) and not *muqaddimāt*. A *muqaddama* is that part of a proof which is *put forward*.

[8] See ALBERT LEJEUNE, *Recherches sur la catoptrique grecque*, Brussels: Académie Royale de Belgique, 1957, pp. 71 ff.

[9] M. NAẒĪF, *al-Ḥasan ibn al-Haytham, buḥūthuhu wa kushūfuhu al-baṣariyya*, 2 vols., Cairo: Fouad I University, 1942–1943. This contains the best and most detailed study of IBN AL-HAYTHAM'S treatment of the reflection-point(s) problem in any language; see vol. II, pp. 487–589. The two best historical accounts of "Alhazen's problem" in a European language are P. BODE, "Die Alhazensche Spiegelaufgabe in ihrer historischen Entwicklung ...", in *Jahresbericht des Physikalischen Vereins zu Frankfurt am Main*, for 1891–1892 (1893), pp. 63–107; and J. A. LOHNE, "Alhazens Spiegelproblem", in *Nordisk matematisk tidskrit*, **18** (1970), pp. 5–35 (with bibliography). For the transmission of IBN AL-HAYTHAM'S problem to the Latin Middle Ages (in so far as it relates to conic

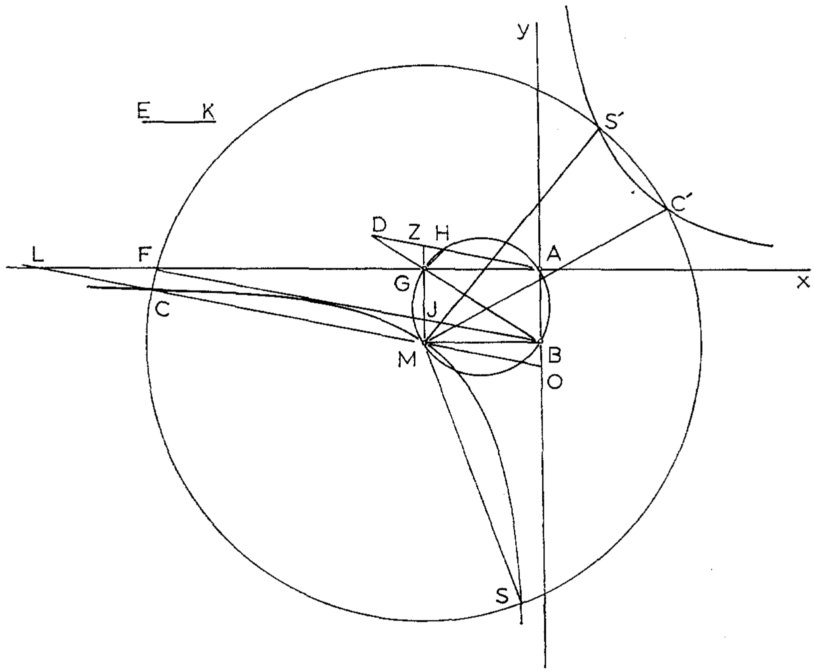

Fig. 2

We are given a point A on the circumference of a circle with diameter BG; and we are required to draw a line that cuts the circumference at a point, like H, and the diameter or its extension at another point, like D, such that DH equals a given line KE.[10]

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sections), see MARSHALL CLAGETT, *Archimedes in the Middle Ages*, vol. IV (A supplement on the medieval Latin traditions of conic sections, 1150–1566), Philadelphia: The American Philosophical Society, 1980, Chapter 1, pp. 3–31.

[10] Or, to phrase the problem differently, it is required to place between the diameter BG (or BG produced) and the circumference of the circle ABG a line equal to KE and verging towards the given point A. This is a particular case of the type of problem known to the Greeks as *neusis* (verging). PAPPUS, in his *Mathematical Collection*, presents several cases of the problem including that in which it is required to place a straight line of a given length between two straight lines given in position and verging towards a given point—a construction which, he tells us, the Greeks had ultimately solved by the use of conic sections. He himself shows a solution by means of the intersection of a hyperbola and a circle. The Greeks used the *neusis* as an intermediate step in the solution of the problem of trisecting an acute rectilineal angle. Their procedure appears to have become known to the Baghdad mathematicians of the ninth century, though not through direct translation of PAPPUS' text. J. P. HOGENDIJK sheds light on the transmission of this Greek method into Arabic, in "How trisections of the angle were transmitted from Greek to Islamic Geometry", *Historia Mathematica*, 8 (1981), pp. 417–38.

It may be noted further that Prop. 8 in the *Liber assumptorum* (attributed to ARCHIMEDES but found only in Arabic) assumes (without proof) a *neusis* construction in which a line segment of given length is to be placed between the circumference of a circle and the

Join AG, AB and produce the mon both sides to form the rectangular axes x and y with A as origin.

Draw GM parallel to AB, and let it cut the circumference of the circle ABG in M.

Through M draw the hyperbola whose asymptotes are the two axes.

Then find the line MC whose product with KE is equal to the square of the diameter BG, *i.e.*

$$MC \cdot KE = \overline{BG}^2,$$

or

$$MC = \frac{\overline{BG}^2}{KE}.$$

The circle about M, with radius MC, will, in general, cut the two branches of the hyperbola in four points—let these be C, S, C', S'.

Join the lines MC, MS, MC', MS'.

Then each of the lines drawn from A parallel to these four lines will be the required line.

For example, line AHD, drawn parallel to MC cuts the circumference at H and the extension of the diameter BG at D, such that $DH = KE$.

In Figure 3 all four parallel lines are shown:

AH_2D_2, parallel to MS, cuts the circumference in H_2

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extension of the circle's diameter, such that the line segment verges towards a given point on the circle's circumference. Similar cases of *neusis* construction occur in ARCHIMEDES' work *On Spirals*, again without proofs. See T. L. HEATH, *The Works of Archimedes*, New York: Dover Publications, Inc. (reprint of 1912 edition), undated, Introduction, ch. V, pp. c–cxxii; *A History of Greek Mathematics*, vol. I (Oxford: The Clarendon Press), pp. 235–41; *A Manual of Greek Mathematics*, New York: Dover Publications, Inc. (reprint of the Oxford edition of 1931), pp. 147–52.

ABŪ SAHL AL-QŪHĪ, who flourished at Baghdad some fifty years before IBN AL-HAYTHAM died (see *Dictionary of Scientific Biography*, XI (1975), pp. 239–41), in a letter to ABŪ ISḤĀQ AL-ṢĀBI' (MS Ayasofya 4832, pp. 133^b–140^a, especially 138^a–139^a) assumes the solution of the following verging problem: to draw from a given point outside a given angle a line that cuts the sides of the angle, such that the intercept between these sides equals a given line. Instead of providing a proof AL-QŪHĪ simply says "We have shown how to do this in many places and it may often happen (*rubba-mā yattafiqu*) that we do not need [for this purpose] to resort to conic sections" (p. 138^b). (J. L. BERGGREN drew my attention to this passage.) It is known that IBN AL-HATHAM was acquainted with at least some of AL-QŪHĪ's works (see, for example, R. RASHED, "La construction de l'heptagone régulier par Ibn al-Haytham," *Journal for the History of Arabic Science*, **3** (1979), p. 341 (French), p. 228 (Arabic)). But the whole question of IBN AL-HAYTHAM's sources remains largely unexplored. That he was well versed in the methods of Greek higher mathematics is clear from several of his writings (including the *Optics*) and from the fact that he felt able to attempt a reconstruction of the lost book VIII of APOLLONIUS' *Conics*. This reconstruction, extant in a unique MS in Turkey (Manisa, Genel 1706, 1^b–25^b; see F. SEZGIN, *Geschichte des arabiscen Schrifttums*, V (Leiden: E. J. Brill, 1974), p. 140), and published in facsimile by NAZIM TERZIOĞLU as *Das achte Buch zu dem "Conica" des Appollonios von Perge*, rekonstruiert von Ibn al-Hayṯam, Istanbul, 1974, is being studied by J. HOGENDIJK of the University of Utrecht.

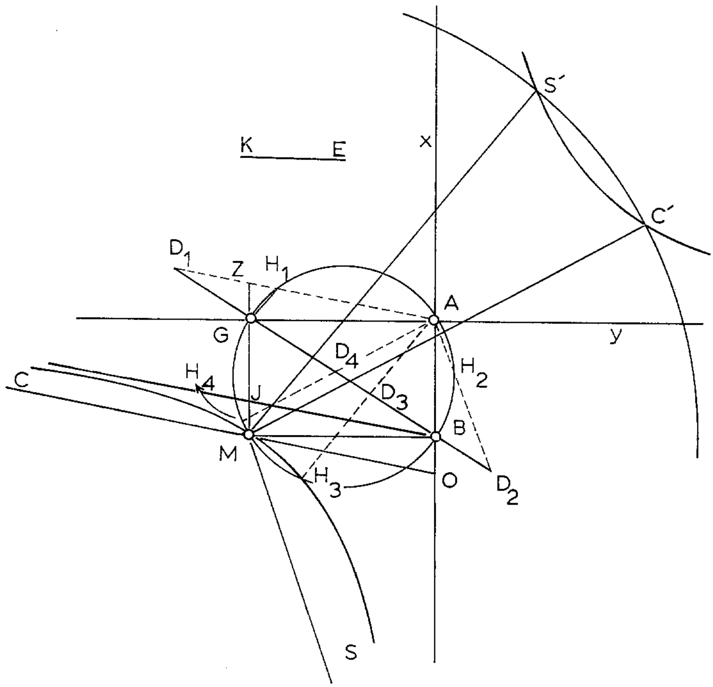

Fig. 3

and GB produced in D_2;

AD_3H_3, parallel to MS', cuts the circumference in H_3

and the diameter in D_3; and

AD_4H_4, parallel to MC', cuts the circumference in H_4 and

the diameter in D_4.

As in the case of AH_1D_1, the portion of each one of these lines between the circumference and the diameter is equal to the given line KE. That is H_2D_2, H_3D_3, H_4D_4 are each equal to KE.

The construction in Figure 3 therefore yields a general solution of our problem. But before we turn to IBN AL-HAYTHAM'S lemmas it should be noted that while the circle with radius MC will always cut the branch of the hyperbola through M in two points, three possibilities exist with regard to the other branch:

(a) the circle may cut it in two points, as in the figure (and this makes it possible to draw *two* lines satisfying the stated condition),

or

(b) the circle may touch that branch at one point (and this allows the construction of one line satisfying the stated condition),

or

(c) the circle may fall short of it altogether (and in this last case the required line cannot be constructed).

All this simply follows from the fact that the radius of the cutting circle, MC, is equal to $\overline{BG}^2/KE$ and therefore depends on KE.

With this picture in mind, IBN AL-HAYTHAM's own procedure should now be easy to follow. As in all of his proofs, the problem is divided into particular cases which are taken up one by one. Figure 4 represents what I shall call case (a) in

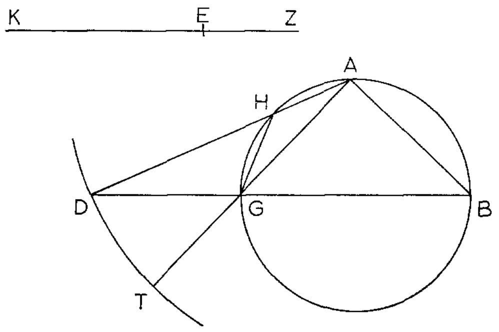

Fig. 4 = Lemma I

the first of the six lemmas: the given point A on the circumference of the circle having the diameter BG lies at the middle of the semi-circle BAG; and we are to draw a line, as AHD, cutting the circumference in H and the extension of BG (in this direction) in D, such that HD is equal to the given line KE.

KE is produced to Z such that

$$KZ \cdot ZE = \overline{AG}^2 \quad (KZ > AG)$$

and AT, equal to KZ, is drawn through G.

The circle about A with radius AT will cut the extension of the diameter BG, say at D,

and line AD will cut arc AG in H.

The required line is HD—which follows from the observation that triangles AGD, AHG are similar.

Case (b) in Lemma I is more complicated; it admits of three sub-cases (Fig. 5). The required line may be tangent to the circle at the given point A (as in 1), or it may cut the circle at a second point H which may lie on arc AG (as in 2), or on arc BA (as in 3).

IBN AL-HAYTHAM provides proofs for all these cases, all based on the construction on the left. It is this construction which should now be described.

TN is a line taken at random.

Having drawn GZ (say in case 1) parallel to BA, the following angles and lines are then constructed:

$$\sphericalangle TNL = \sphericalangle DGA,$$

$$\sphericalangle TNM = \sphericalangle DGZ,$$

$$\text{line } MT \,//\, \text{line } LN,$$

and

$$\text{line } TQ \,//\, \text{line } MN.$$

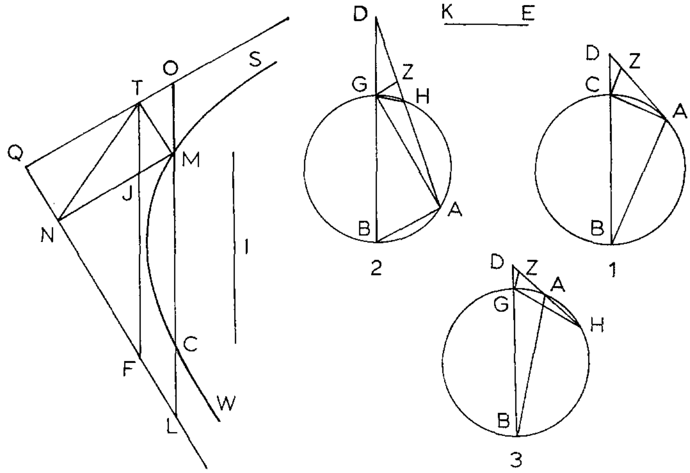

Fig. 5 = Lemma I

Referring to APOLLONIUS' *Conics*, Bk. II, Prop. 4, IBN AL-HAYTHAM then draws the branch of the hyperbola through M, with QT, QL as asymptotes (the similarity with Fig. 2 is apparent).

On the branch SMW, take a point C, such that

$$\frac{MC}{TN} = \frac{BG}{KE}.$$

Referring again to APOLLONIUS' *Conics*, Bk. II. Prop. 8, IBN AL-HAYTHAM states that the extension of MC on both sides will cut the asymptotes in points O and L, such that

$$OM = LC.$$

Draw TF parallel to OL, cutting NM in J.

Since surface $TMLF$ is a parallelogram, and so also is surface $TOMJ$, it follows that

$$MC = JF,$$

and therefore

$$\frac{JF}{TN} = \frac{BG}{KE}.$$

If AZ is now drawn at an angle

$$GAZ = NFT,$$

it will cut BG produced—say at D.

IBN AL-HAYTHAM shows, with reference to each of the three cases separately, that line AD will meet the cricumference at H and the extension of the diameter at D, such that $HD = KE$.

The difficulty with IBN AL-HAYTHAM'S approach, as compared with that of seventeenth-century mathematicians, becomes immediately apparent when we note that Lemma I, consisting of four particular cases, is designed to take care of only one of the four lines in our reference Figure 3, namely line AD_1 which cuts the extension of the diameter *BG* on the side of *G*. IBN AL-HAYTHAM says nothing about line AD_2, cutting the extension of the diameter on the other side. But he provides a second lemma for the construction of lines AH_3, AH_4 which intersect the diameter itself. A brief look at this lemma will also be instructive.

In Figure 6, constructed from the text of Lemma II, *A* (in the right-hand figure) is the given point on the circumference of the circle with diameter *BG*; and we are to draw from *A* a line that cuts *BG* and the circumference in two points, such as *E*, *D*, so that *DE* is equal to the given line *HZ*.

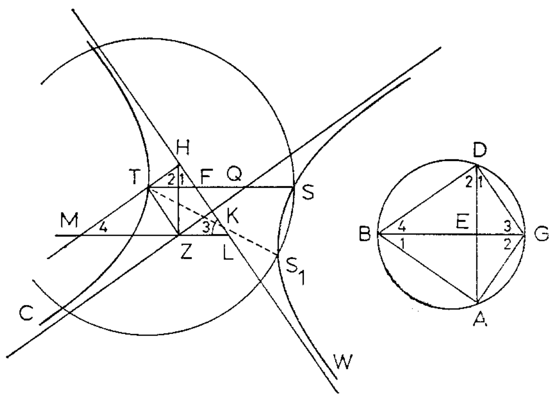

Fig. 6 = Lemma II

Having drawn *AB*, *AG*, IBN AL-HAYTHAM constructs angles *H*1 and *H*2 on either side of *HZ*, equal to angles *B*1 and *G*2 respectively. He completes the parallelogram *HKZT*, and draws through *T* the branch of the hyperbola with *KH* and *KZ* as asymptotes. Then, with *T* as centre and a radius equal to *BG*, he draws a circle that, according to his own explicit remarks, may or may not cut the opposite branch of the hyperbola. His text, however, is concerned with the case in which a meeting of the circle and that branch does take place, for example, at point *S*.

He joins *TS*, cutting the asymptotes at *F* and *Q*; and, through point *Z*, he draws *LZM* parallel to *TS*, and, like *TS*, cutting both asymptotes. *LZM* will cut the extension of *HT*, say in *M*. Finally, he draws *GD* at an angle with *BG* equal to *MLH*, and joins *BD*.

Considerations of the similar triangles indicated in the figure entail the equality of *DE* to the given line *HZ*.

The corresponding figures in RISNER and in KAMĀL AL-DĪN, inadequately and inexactly drawn, do not include the circle through *S* or the discontinuous line TS_1. This seems to reflect IBN AL-HAYTHAM's remarks just referred to. He states that from *T* on one branch of the hyperbola, it may not be possible to draw more than one line that reaches the other branch. This, of course, would be the case when the circle touches that other branch at a point. He also notes that in some cases two such lines may be drawn (as in our Fig. 6), and, further, that for the construction of the required line to be at all possible, it is necessary that *BG*, equal to the radius of the circle, must not, in his words, "be shorter than the shortest line that can be drawn from *T* to section *SW*".[11] As to the question of how this shortest line should be determined he refers the reader to Propositions 34 and 61 of Bk. V of the *Conics*—a correct reference which is omitted in RISNER.

So much for that part of IBN AL-HAYTHAM's proof. The next steps are not difficult to follow, but IBN AL-HAYTHAM's method of procedure remains the same. Lemmas III and VI are particular cases of one problem, and they establish their conclusions by reference to Lemmas I and II respectively.

Figures 7.1 and 7.2 are drawn from the text of Lemma III. In the triangle *ABG*, *B* is a right angle, and *D* a point given on *BG* (as in Fig. 7.1) or on its exten-

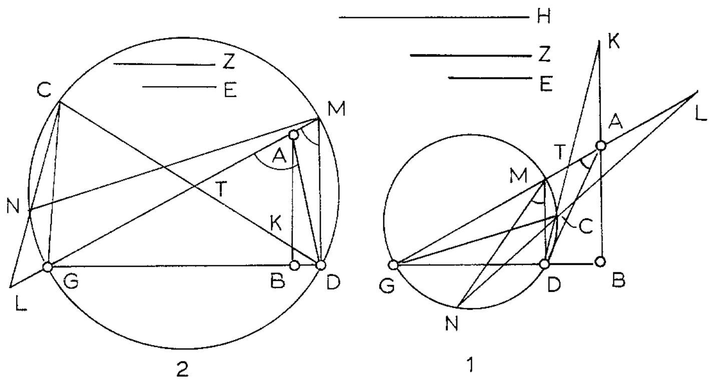

Fig. 7 = Lemma III

sion toward *B* (as in Fig. 7.2). It is required to draw from *D* a line that cuts the hypotenuse in a point, as *T*, and *AB* or its extension in another point, as *K*, such that

TK is to *TG* in a given ratio $(E:Z)$.

From now on it will be easier to concentrate on Figure 7.1. Join *AD*; draw *DM* parallel to *BA* and describe the circle about the right-angled triangle *MDG*, which will have *GM* as diameter.

Construct angle *DMN* equal to angle *DAG*.

N will be on arc *DG* (Fig. 7.1), or on arc *MG* (Fig. 7.2).

Three more steps complete the figure. First, construct a line *H*, such that

$$\frac{AD}{H} = \frac{E}{Z} \text{(the given ratio)}.$$

Then, *applying Lemma I*, draw from *N* the line *NCL*, so that *CL*, the distance between the line's intersection with the circumference and the extension of diameter *MG*, is equal to *H*.

Now join *DC* and produce it in a straight line: it will cut *LM*, say in *T*. And join *GC*.

[11] See below, p. 318.

Ibn al-Haytham shows that DT produced will cut BA produced (in Fig. 7.1) in a point K such that

$$\sphericalangle AKT = \sphericalangle TDM = \sphericalangle TGC.$$

Finally, from the similarity of triangles AKT and CGT, and also triangles LCT and ADT, it follows that

$$\frac{KT}{TG} = \frac{AT}{TC} = \frac{AD}{CL} = \frac{AD}{H} = \frac{E}{Z}, \quad \text{Q.E.F.}$$

The remaining case in this problem, represented by Lemma VI, relates to Figure 1.2, *i.e.* the auxiliary figure for the construction of the reflection-point on the surface of a spherical convex mirror.

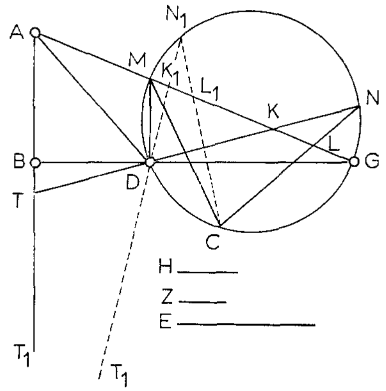

Fig. 8 = Lemma VI

Here (Fig. 8) from point D on side BG of the right-angled triangle ABG, we are to draw a line that cuts the hypotenuse in K and the extension of AB in T, such that

$$\frac{KT}{KG} = \frac{E}{Z} = \text{a given ratio}.$$

This Ibn al-Haytham achieves on the basis of Lemma II which allows him to draw line CLN, cutting the diameter of the circle about MDG in L and the circumference in N, such that

$$LN = H,$$

where H is determined by

$$\frac{AD}{H} = \frac{E}{Z}, \text{the given ratio}.$$

We know, however, that it may be possible in this case to draw a second line, as CL_1N_1, which satisfies the stated condition, namely such that $L_1N_1 = H$. If that is the case, then, in addition to line $NKDT$, another line $N_1K_1DT_1$ can be drawn so that T_1K_1 is to K_1G as E is to Z. Again the figures in Risner and in KAMĀL AL-DĪN do not show the discontinuous lines in our figure. But IBN AL-HAYTHAM'S text is explicit. This is what he says:

> "... it was shown earlier [i.e. in Lemma II] that there issue from point C two lines such that the segment of each of them that lies between the circle and the diameter [here segments LN and L_1N_1] will be equal to the given line [H]. Thus *if* two such lines are drawn from C, then there will issue from point D two lines in the given ratio; but the two angles produced at point G will be unequal ... [he means the angles made by TG or T_1G with AG]."[12]

This concluding comment is paraphrased in RISNER without the reference to the unequal angles at G.[13]

We come now to an important step in IBN AL-HAYTHAM'S procedure, represented by Lemma IV.

In the plane of the circle with radius BG (Fig. 9.1), two points, say D and E, are given: and we are to find on the circumference of the circle a point A such that the tangent at A (AH in the figure) bisects the angle contained by AD and AE.

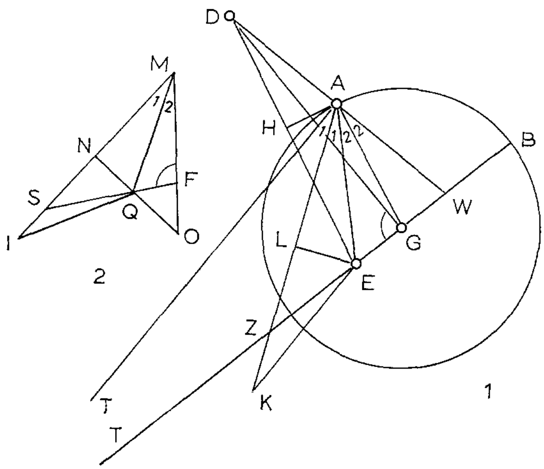

Fig. 9 = Lemma IV

Rather than summarize the proof, which *is* long, I shall be concerned to point out some features of it. The proof makes use of Figure 9.2 which is but case 2 of Lemma III (see Fig. 8.2), where from D on the extension of GB in the right triangle ABG, a line DKT is to be drawn, so that TK is to TG in a given ratio.

Similarly, to go back to Figure 9.2, SQF is drawn so that QF to FM is in a given ratio (—in this case, EG to GB in Fig. 9.1).

Now it is clear that the condition stated in this Lemma (that the tangent at A bisects angle DAE) is a particular case of a more general condition that can be stated by requiring that the tangent AH should make equal angles with AD and

[12] See below, p. 324. Emphasis added.

[13] *Opticae thesaurus. Alhazeni libri septem*, sec. 38, p. 150.

AE, without necessarily bisecting the angle contained by these two lines. Starting from this observation, NAẒĪF provides a generalized construction for Lemma IV that yields four points satisfying the more general condition.[14] This, in turn, yields a general solution of the problem of finding the reflection-point on the surface of a spherical concave mirror.

Figure 10 is an illustration of NAẒĪF's construction, where A and B are the positions of the eye and the visible object respectively, and P_1, P_2, P_3 and P_4 are reflection-points on the surface of the concave mirror with radius GM.

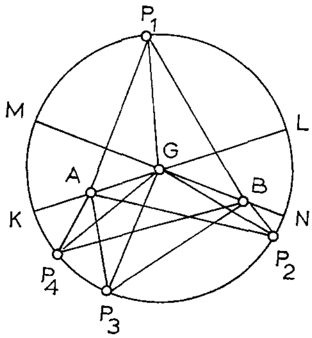

Fig. 10

NAẒĪF's construction is valid inasmuch as it is based on Lemmas III and VI which together comprize four possible cases. It does not, however, reflect IBN AL-HAYTHAM's intention, which (as NAẒĪF also points out)[15] is obviously to propose a particular construction (in which one of the two given points lies outside the circle) with a particular application in mind.

A similar observation applies to Lemma V. In Figure 11, E is a point given

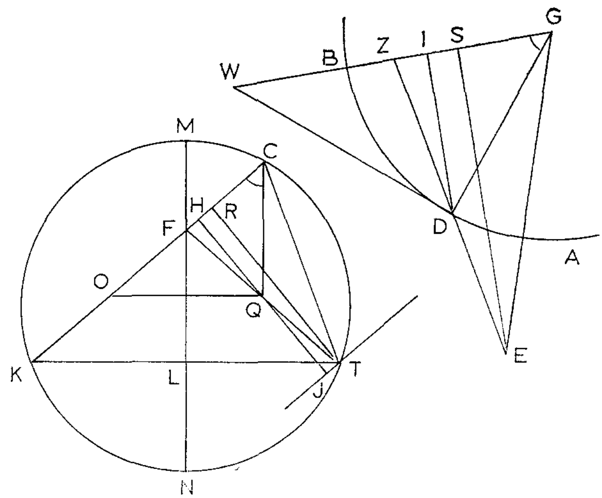

Fig. 11 = Lemma V

[14] NAẒĪF, *op. cit.*, vol. II, pp. 515–27.

[15] *Ibid.*, pp. 524–7.

outside the circle with radius BG; and it is required to draw from E a line that cuts the circumference in a point like D and the diameter in a point like Z, so that

$$DZ = ZG.$$

Having drawn the perpendicular ES, IBN AL-HAYTHAM takes a line $KT = ES$ on which he describes the segment of a circle that admits an angle equal to BGE.

Then, having drawn the diameter MN through the middle of KT, he constructs line KFC, such that

$$FC = \frac{1}{2} BG.$$

This construction relies of course on Lemma II. But since the diameter MN is greater than the radius of the given circle BG, four lines can generally be drawn that satisfy the stated condition. However, IBN AL-HAYTHAM neither considers nor refers to any line other than KFC. Nor does he consider or refer to the case in which E lies inside the given circle.

So here again IBN AL-HAYTHAM is concerned with a particular case to be applied later to a particular construction.

This can be clearly illustrated by IBN AL-HAYTHAM'S own construction for the reflection-point on the surface of a spherical convex mirror (Fig. 1). Here the conditions he lays down for drawing line SFQ (in particular, that angle SNQ must be obtuse) is equivalent to asserting that A and B (the two points related as object and eye) must be such that the line joining them neither cuts nor is tangent to the circle. If this condition does not obtain, no reflection from the convex side of the mirror will take place. (His investigation of this type of mirror is completed by a *reductio ad absurdum* proof that shows that no more than one reflection-point is possible.)

How, then, does IBN AL-HAYTHAM find the reflection-point (or points) on the surface of a spherical concave mirror? He enumerates the special cases and deals with them one by one. The two points related as object and eye may lie on the diameter of the mirror (or on its extension) at equal or unequal distances from the centre of the mirror. Or they may lie on different diameters, their distances from the centre being equal or unequal. IBN AL-HAYTHAM'S piecemeal treatment of these cases, in which he applies his lemmas as required, makes for an even longer story than the one I have just summarized. But adding all these cases together we obtain a general solution of "Alhazen's problem" in HUYGENS' restricted sense. Long or not, this was an impressive achievement. But the historian's job is not completed before other investigations have been carried out. We still, for example, have to identify IBN AL-HAYTHAM'S sources and find a detailed explanation for the character of his approach.

The preceding account had two limited aims: to give an accurate, though abbreviated, description of IBN AL-HAYTHAM'S procedure by providing exact figures that correspond to his own text, and to point out certain features of his proof that must be borne in mind in studying their character, their influence, and the reactions (and misunderstandings) they have given rise to. These two aims must be fully realized before we can put ourselves in a position to achieve an exact assessment of IBN AL-HAYTHAM's contribution, or make meaningful comparisons between his performance and that of later mathematicians.

II

Translation of Ibn al-Haytham's Lemmas[16]

[Lemma I: Figures 4 and 5]

Let circle *ABG* [Fig. 4], with diameter *GB*, be known [*maᶜlūma*]; let *GB* be produced on the side of *G*; let line *KE* be given [*mafrūḍ*] and let point *A* be given on the circumference of the circle. We wish to draw from *A* a line, as *AHD*, so that the part of it that lies between the diameter and the circle—such as *HD*—is equal to line *KE*.

Now arcs *BA*, *AG* are either equal to one another or not.

Let them be equal. We join lines *BA*, *AG*, and make the product of *KZ* and *ZE* equal to the square of *AG*. Line *KZ* will then be greater than line *AG*.

Draw *AG* and make *AT* equal to *KZ*;

with *A* as centre and with distance *AT*, draw an arc of a circle: it will always cut line *GD*—let it cut it at *D*.

Join *AD*: the line *AD* will be equal to line *KZ*.

AD will always cut arc *AG*, since the line drawn tangentially from *A* will be parallel to *GB*; for the line from point *A* joined to the circle's centre will be perpendicular to line *GB*, because of the equality of arcs *AB*, *AG*. Therefore line *AD* will cut arc *AG*—let it cut it at point *H*.

Join *GH*.

Angles *AHG*, *ABG* will together be equal to two right angles.

But angle *ABG* is equal to angle *AGB*;

therefore angle *AHG* is equal to angle *AGD*;

therefore triangle *ADG* is similar to triangle *AGH*.

It follows that the ratio of *DA* to *AG* is as the ratio of *GA* to *AH*, and, therefore, the product of *DA* and *AH* is equal to the square of *AG*.

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16 The following translation is made from my (as yet unpublished) edition of the Arabic text in Book V of *Kitāb al-Manāẓir*. Book V survives in three MSS which are all preserved in Istanbul libraries: Fatih 3215, fols. 138^a–332^b, dated Jumādā II, 636/A.D. 1239; Ayasofya 2448, fols. 386^b–508^a, dated A.H. 869/A.D. 1464–1465; and Köprülü 952, fols 2^{a-b}, 74^a–81^b, 89^a–107^b, 134^a–135^b, dating probably from the 14th century A.D. All geometrical diagrams for Book V are missing from the Fatih and Ayasofya MSS. The Köprülü MS is incomplete but has the diagrams associated with the part of the text which it includes. I have made use of KAMĀL AL-DĪN's *Tanqīḥ* and of RISNER's edition of the medieval Latin version of *Kitāb al-Manāẓir*, both of which include the diagrams but not always accurately drawn.

In transliterating the Arabic I have used *C* for *ṣād*, *J* for *shīn* and *t* for *ṭā*'. All other transliterations are standard in recent literature.

But the product of *KZ* and *ZE* is equal to the square of *AG*;
therefore the product of *DA* and *AH* is equal to the product of *KZ* and *ZE*.
And *DA* is equal to *KZ*; therefore *AH* is equal to *ZE*.
It remains that line *HD* is equal to line *KE*.
And that is what we wished to do.

Now let arcs *BA*, *AG* be unequal [Fig. 5]. We join lines *BA*, *AG*, and draw *GZ* parallel to *BA*. Take a given line at random; let it be *TN*. Make angle *TNL* equal to angle *DGA*, and angle *TNM* equal to angle *DGZ*;
produce line *LN* on the side of *N* to *Q*, and draw line *MT* parallel to line *NL*;
further, draw line *TQ* parallel to *NM*, and produce *QT* on the side of *T* to *O*.
Then, through *M*, we draw the hyperbola of which lines *OQ*, *QL* are asymptotes (as has been shown in Proposition 4 in Book II of the *Conics* of Apollonius) — and let it be section *SMW*;
make the ratio of line *I* to line *TN* as the ratio of line *BG* to line *KE*;
draw in section *SMW* line *MC* equal to line *I*, and produce *MC* on both sides;
it will meet lines *LQ*, *QO* (as has been shown in Proposition 8 in Book II of the *Conics*)—and let it meet them in points *L*, *O*.
Then lines *OM*, *LC* will be equal (as has been shown also in Proposition 8 of the said Book).
Draw from point *T* line *TF* parallel to line *OL*, and let it cut line *NM* in point *J*.
Thus, surface *LMTF* being a parallelogram, line *LM* will be equal to line *FT*.
But *LM* is equal to *CO*,
therefore *CO* is equal to *TF*;
and *MO* is equal to *JT*, because surface *JO* is a parallelogram,
it remains that *FJ* is equal to *CM*;
and *CM* is equal to *I*,
therefore line *FJ* is equal to line *I*;
and it follows that the ratio of line *FJ* to line *TN* is as the ratio of *BG* to *KE*.
On line *GA* and at point *A* draw angle *GAZ* equal to angle *NFT*.
This line, i.e. line *AZ*, will meet line *GD*, because the angles at points *A*, *G* are equal to the angles at points *F*, *N*—let it meet *GD* at *D*.
Now since angles *AGD*, *ZGD* are equal to angles *FNT*, *JNT*,
and angle *GAD* is equal to angle *NFT*,
triangles *AGZ*, *ZGD*, *AGD* are similar to triangles *FNJ*, *JNT*, *FNT*,
and, therefore, as *ZA* is to *AG* so is *JF* to *FN*,
and, as *AG* is to *GD*, so is *FN* to *NT*;
therefore as *AZ* is to *GD* so is *FJ* to *NT*.
But *FJ* is equal to *I*, and as *I* is to *TN* so is *BG* to *KE*,
therefore as *AZ* is to *GD* so is *BG* to *KE*.

And since line *AD* meets *BD* outside the circle on the side of *G*, line *DA* will either touch the circle at point *A* [Fig. 5.1], or it will cut arc *AG* [Fig. 5.2], or else cut arc *AB* [Fig. 5.3].
For, if arc *AG* is smaller than arc *AB* [Fig. 5.1], then the tangent drawn from *A* will meet the diameter *BG* on the side of *G*, and the line drawn from *A* parallel to diameter *BG* will cut arc *AB*; and, therefore, the lines which are drawn from *A*

and which meet *GD* above the tangent will cut the part of arc *AB* that is cut off by the parallel line. Further, the lines which are drawn from point *A* and which meet *GD* below the tangent will cut arc *AG*.

Now let arc *AG* be greater than arc *AB* [Fig. 5.2]; then every line drawn from *A*, meeting *BG* outside the circle on the side of *G*, will always cut [arc] *AG*.
For the tangent drawn from *A* will meet *BG* on the side of *B*,
and the line drawn from *A* parallel to the diameter *BG* will cut arc *AG*;
from which it follows (if arc *AG* is greater than arc *AB*) that all lines drawn from *A* so as to meet *BG* outside the circle on the side of *G* will cut arc *AG*.
Thus line *AD* will either touch the circle at *A* (as in the First Figure), or cut arc *AG* (as in the Second Figure), or else cut arc *AB* (as in the Third Figure).

[And, first,] let it be tangent [to the circle, as in Fig. 5.1].
Then angle *GAD* is equal to angle *ABG*,
and angle *ZGD* is equal to angle *ABG*,
therefore angle *ZGD* is equal to angle *GAD*.
Therefore the product of *AD* and *DZ* is equal to the square of *GD*;
and the product of *BD* and *DG* is equal to the square of *AD* (because *AD* is a tangent);
it remains that the product of *DA* and *AZ* is equal to the product of *BG* and *GD*.
Therefore as *AZ* is to *GD*, so is *BG* to *DA*;
but *AZ* to *GD* was shown to be as *BG* is to *KE*;
therefore as *BG* is to *KE* so is *BG* to *DA*;
and, therefore, line *DA* is equal to line *KE*.

Now let line *AD* cut arc *AG*, say at point *H* [Fig. 5.2].
Join *GH*.
Angle *AHG* will then together with angle *ABG* be equal to two right angles.
Therefore angle *GHZ* is equal to angle *ABG*;
and angle *ZGD* is equal to angle *ABG*; therefore angle *GHZ* is equal to angle *ZGD*;
therefore the product of *HD* and *DZ* is equal to the square of *GD*;
and the product of *AD* and *DH* is equal to the product of *BD* and *DG*;
it remains that the product of *HD* and *AZ* is equal to the product of *BG* and *DG*.
Therefore as *AZ* is to *GD* so is *BG* to *HD*;
but *AZ* to *GD* was [shown to be] as *BG* is to *KE*; therefore as *BG* is to *HD* so is *BG* to *KE*;
therefore line *HD* is equal to line *KE*.
Now let line *AD* cut arc *AB*, say at point *H* [Fig. 5.3].
Join *HG*.
Thus angle *GHA* is equal to angle *GBA*;
and angle *ZGD* is equal to angle *GBA*;
therefore angle *GHD* is equal to angle *DGZ*.
Therefore the product of *HD* and *DZ* is equal to the square of *GD*;

but the product of HD and AD is equal to the product of BD and DG;
it remains that the product of HD and AZ is equal to the product of BG and GD.
Therefore as AZ is to GD, so is BG to HD;
but AZ is to GD as BG is to KE;
therefore as BG is to HD so is BG to KE;
therefore line HD is equal to line KE.

We have thus shown in all cases how to draw from A a line that meets the diameter BG outside the circle on the side of G, so that the part of the line that lies between the circle and the diameter is equal to line KE
And that is what we wished to do.

[Lemma II: Figure 6]

Again, let [points] A, B, G be on the circumference of a circle; let BG be a diameter, and let line ZH be given; we wish to draw from A a line that cuts diameter BG and carries through to the circle, so that the part of it that lies between the circle and the diameter will be equal to line ZH.
Join lines AB, AG; and on line ZH and at point H construct angle ZHK equal to angle ABG, and angle ZHT equal to angle AGB;
from Z draw line ZT parallel to line KH, and ZK parallel to TH;
thus surface TK will be a parallelogram.
Draw through point T the hyperbola of which lines HK, KZ are asymptotes—let it be section TC, and let the opposite section be WS;
produce lines HK, ZK on the side of K to L, F, and with T as centre, and with a distance equal to diameter BG, describe a circle, and let this circle meet section WS at point S.
This circle will meet section WS if BG is not smaller than the shortest line that can be drawn from point T to section WS.
As to which is the shortest line that can be drawn from T to section WS, this has been shown in Propositions 34 and 61 in Book V of Apollonius' *Conics*.
Thus the circle described about T with distance BG, if it meets the section, will either touch it at one point or cut it in two points.
If it touches the circle, then only one line equal to BG can be drawn from point T to section WS.
But if the circle cuts the section in two points, then only two lines equal to BG can be drawn from point T to section WS.
Thus point S is either the point of tangency or one of the two points of intersection.
Join line TS; it will be equal to BG.
Line TS will thus cut lines HK, KQ—let it cut HK in point F, and KQ in point Q;
draw from Z a line parallel to TS, which will cut lines HK, HT, since line TS cuts these two lines—let that be line LZM;
thus ZM will be equal to TQ, because surface MQ is a parallelogram.

Now since *CT*, *WS* are opposite sections,
and *TS* cuts their asymptotes,
line *TF* will be equal to line *QS* (as is shown in Proposition 16[17] in Book II of the *Conics*).

And *TF* is equal to line *ZL*, because surface *LT* is a parallelogram,
therefore *ZL* is equal to *QS*;
and *ZM* is equal to *TQ*,
therefore *LM* is equal to *TS*;
and *TS* is equal to *BG*,
therefore *LM* is equal to *BG*.

We further construct on line *BG*, at point *G*, an angle *BGD* equal to angle *MLH*.

Angle *MLH* will be acute because angle *LHM* is right, being equal to *ABG* and *AGB*.

Line *GD* will therefore fall inside the circle—let it cut the circle at point *D*.
Join *BD*, *AD*, and let *AD* cut *BG* at point *E*.
Angle *GDB* will be a right angle, equal to *LHM*,
and angle *BDE* will be qeual to angle *BGA* which is equal to angle *ZHM*,
and angle *GBD* will be equal to angle *LMH*.
Thus triangle *BGD* will be similar to triangle *LMH*,
and triangle *DEB* will be similar to triangle *HZM*.
Therefore as *GB* is to *BD*, so is *LM* to *MH*;
and *BD* is to *DE* as *MH* is to *HZ*,
therefore as *GB* is to *ED* so is *LM* to *ZH*;
but *LM* is equal to *BG*,
therefore *DE* is equal to *ZH*.

We have thus drawn from point *A* line *AED* so that line *ED* is equal to line *ZH*.

And that is what we wished to do.

But if two lines equal to *BG* go from point *T* to section *WS*, then there will go from point *Z* to lines *KH*, *HT* two lines equal to line *BG*, producing between them and line *HK* two unequal angles.

Then if two angles equal to those angles are constructed on line *BG* at point *G*, two points will be produced on arc *BG*.

And if two lines are joined between them and point *A*, there will be cut off from each of these lines between arc *BDG* and diameter *BG* a line equal to *ZH*—this being shown by the demonstration we mentioned.

Further, if line *BG* is equal to the shortest line that can be drawn from point *T*

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[17] All three MSS have "11" instead of "16", the correct number of the proposition in Bk. II of the *Conics* both in Heiberg's edition of the Greek text and in the Arabic copy in Ibn al-Haytham's own hand (MS Ayasofya 2762). The wrong number "11" is written out in words in the Köprülü MS, and in the *abjad* notation in the Ayasofya and Fatih MSS.

to section *WS*, then only one line can be drawn from *A* to arc *BDG* so that the segment between the arc and line *BG* is equal to *ZH*.

If *BG* is greater than the shortest line, then there will go from *A* to arc *BDG* two lines in each of which the segment between the arc and the diameter will be equal to line *ZH*.

No more than two lines can be drawn from *A* to arc *BDG* so that the segment between the arc and the diameter will be equal to *ZH*. For the circle about centre *T* cannot cut section *WS* at more than two points, the centre of the circle being outside the section.

And, further, if *BG* is smaller than the shortest line, then a line cannot be drawn from *A* to arc *BDG*, so that the segment between the arc and the diameter is equal to *ZH*.

This construction is, therefore, either impossible, or it can be carried out once, or twice, but not more.

And that is what we wished to do.

[Lemma III: Figure 7]

Again, in triangle *ABG* let angle *B* be right; let *D* be given on line *BG*; and let the ratio of *E* to *Z* be known; we wish to draw from *D* a line like *DTK* so that
the ratio of *TK* to *TG*
is as the ratio of *E* to *Z*.

Join *DA*, and draw *DM* parallel to *BA*;
and on triangle *DMG* describe circle *DMG*; *MG* will be a diameter of the circle because *MDG* is a right angle.
Draw angle *DMN* equal to angle *DAG*;
MN will then cut angle *DMG* and, therefore, will cut arc *DG* (as in the First Figure),
or cut arc *MG* (as in the Second Figure);
let it cut [either] arc in point *N*.

Let the ratio of line *AD* to line *H* be as the ratio of *E* to *Z*;
and from *N* draw line *NCL* so that *CL* will be equal to *H* (as was shown earlier);
then join *DC* and produce it in a straight line—it will cut *LM*, say in point *T*;
and join *GC*.

Angle *GCD* will then be equal to angle *GMD*, and, therefore, equal to angle *GAB*,
therefore angle *GCT* is equal to angle *TAK*;
but angle *CTG* is equal to angle *ATK*;
therefore if line *CT* is produced in a straight line (as in the First Figure), it will meet line *AK* at an angle equal to angle *TGC*.
Produce *CT* and let it meet *AK* at *K*.
Then triangle *AKT* will be similar to triangle *CGT* (in both Figures):
therefore as *AT* is to *TC*, so is *KT* to *TG*.

Again, angle *DCN* is equal to angle *DMN*,
and angle *DMN* is equal to angle *DAT*,
therefore angle *LCT* is equal to angle *DAT*.
And triangle *LCT* is similar to triangle *ADT*,
therefore as *AT* is to *TC*, so is *AD* to *LC*.
And *LC* is equal to *H*,
therefore as *AT* is to *TC*, so is *AD* to *H*.
But *AD* is to *H* as *E* is to *Z*,
therefore as *AT* is to *TC*, so is *E* to *Z*.
And *AT* is to *TC* as *KT* is to *TG*,
therefore as *KT* is to *TG*, so is *E* to *Z*.
And that is what we wished to do.

[Lemma IV: Figure 9]

Again, let circle *AB*, with centre *G*, be given, and let *D*, *E* be two given points; we wish to draw from *E*, *D*, two lines like *EA*, *DA*, so that a line drawn tangentially to the circle, such as *AH*, will bisect angle *EAD*.
Join *GD*, *GE*, *ED*; and produce *EG* in a straight line to *B*.
Take any line at random, say *MI*, and divide it at *S*, so that
as *IS* is to *SM*, so is *EG* to *GD*;
then bisect line [*IM*] in *N*, and draw *NO* perpendicular to it;
make angle *NMO* equal to half of angle *DGB*,
and from *S* draw line *SQF*, so that
as *QF* is to *FM*, so is *EG* to *GB*;
and make angle *EGA* equal to angle *SFM*;
and join *EA*, *QM*;
then triangles *EAG*, *QMF* will be similar.

Make angle *EAZ* equal to angle *QMS*;
thus angle *ZAG* will be equal to angle *SMO* which is equal to half of angle *DGB*.
Produce *AZ* on the side of *Z*, and make the ratio of
AZ to *ZK* equal to the ratio of *MS* to *SI*, which is the same as the ratio of *DG* to *GE*.
Join *EK*, *QI*, and draw the perpendicular *EL* [to *AK*].
Thus the angles at points *A*, *E*, *K*, *Z*, *L* will be equal to the angles at points *M*, *Q*, *I*, *S*, *N*, and, therefore, the triangles will be similar.
Therefore *AL* will be equal to *LK*, and *AE* equal to *EK*,
and the ratio of *KZ* to *ZA* will be as the ratio of *IS* to *SM*, which is the same as *EG* is to *GD*.

Draw *AT* parallel to line *EK*.
Therefore angle *TAZ* will be equal to *ZAE*,
and as *EA* is to *AT*,

so will be *EZ* to *ZT*,
and *KZ* to *ZA*,
which is the same as *EG* is to *GD*.

Now make angle *GAW* equal to angle *GAE*.
Therefore angle *WAT* will be double of angle *GAZ*, which is equal to angle *FMN*, which is half of angle *DGB*;
therefore angle *WAT* will be equal to angle *DGW*;
therefore line *WA* will meet line *GD*—if line *AW* meets line *GB*,
and the triangle cut off by line *WA* produced will be similar to triangle *WAT*.
I say, then, that line *WA* will meet line *GD* at point *D*.
For, as *EG* is to *GD*, so is *EA* to *AT*;
and *EA* to *AT* is compounded of *EA* to *AW* and *WA* to *AT*;
therefore *EG* to *GD* is compounded of *EA* to *AW* and *WA* to *AT*.
And as *EA* to *AW*, so is *EG* to *GW*, because the angles at *A* are equal;
and as *WA* is to *AT*, so is *WG* to the line cut off by *WA* from line *GD*;
therefore the ratio of *EG* to *GD* is compounded of *EG* to *GW* and *GW* to the line cut off by *WA* from line *GD*.
But *EG* to *GD* is compounded of *EG* to *GW* and *GW* to *GD*,
therefore *GD* is the line cut off by *WA* and *GD*;
and thus line *WA* will go through to point *D*;
and, therefore, angle *TAD* will be equal to angle *EGD*.

Now make angle *GAH* right.
Then angle *ZAH* will be half of *EGD*, because angle *ZAG* is half of angle *DGW*.
Thus angle *ZAH* is half of angle *TAD*,
and angle *ZAE* is half of angle *TAE*,
therefore angle *EAH* is half of angle *EAD*.

But if line *AW* is parallel to line *GE*, then angle *EGA* will be equal to angle *GAE*;
therefore line *AE* will be equal to line *EG*.
But the angle next to angle *WAT* is equal to angle *TGD*,
and the angle at the intersection of *WA* with *GD* will be equal to angle *TGD*, because they are alternate angles,
therefore line *TA* will be equal to the line cut off by *WA* from line *GD*;
and line *EA* is equal to line *EG*;
therefore as *EA* is to *AT*, so is *EG* to the line cut off by *WA* from *GD*;
but *EA* is to *AT* as *EG* is to *GD*;
therefore the line cut off by *WA* from *GD* is the same as line *GD*;
therefore angle *TAD* will be equal to angle *EGD*.
And angle *ZAH* is half of angle *EGD*,
therefore angle *ZAH* is half of angle *TAD*;
but angle *ZAE* is half of angle *TAE*,
therefore angle *EAH* will in all cases be equal to half of angle *EAD*.
And that is what we wished to prove.

[Lemma V: Figure 11]

Again, let circle *AB* be given, with centre *G* and diameter [*sic*] *GB*, and let point *E* be given outside the circle, and we wish to draw from *E* a line, as *EDZ*, so that *DZ* will be equal to *ZG*.

Join *EG*, and from *E* draw *ES* perpendicular to line *GB*;

and make line *TK* equal to line *ES*;

on line *TK* describe the segment of a circle that admits angle *EGB*, and let it be segment *TMK*, and complete the circle;

bisect *TK* at *L*, and draw *LM* perpendicular to *TK* and carry it through to *N*;

MN will then be a diameter of the circle.

From point *K* draw line *KFC* so that line *CF* will be equal to half of line *GB*.

Join *TF*—it will be equal to *FK*.

Draw *CQ* parallel to *FN*, and *QO* parallel to *KL*;

angle *CQO* will then be a right angle, and *QF* will be equal to *FO*, because *TF* is equal to *FK*.

Then, since angle *CQO* is right and line *QF* is equal to line *FO*,

line *QF* will be equal to *FC*, and *FC* to *FO*.

Construct angle *BGD* equal to angle *KCQ*; join *ED* and carry it through to *Z*.

I say, then, that *DZ* is equal to *ZG*.

Demonstration:

From point *D* draw the perpendicular *DI*, and construct the right angle *GDW*:

line *DW* will then meet *GB*, because angle *DGZ* is acute because it is equal to angle *OCQ*—let them meet at *W*.

Join *TC*, and from *Q* draw the perpendicular *QH*,

draw *TJ* parallel to *CH*, and produce *HQ* to meet it, say at point *J*.

Draw the perpendicular *TR*: it will be equal to *JH*.

Then, since *CF* is half of *GB*, *CO* will be equal to *GD*;

and *TK* is equal to *ES*;

therefore as *TK* is to *CO*, so is *ES* to *GD*.

But as *GD* is to *DI*, so is *GW* to *WD*,

and as *GW* is to *WD*, so is *CO* to *OQ*,

therefore as *ES* is to *DI*, so is *TK* to *QO*, which is the same ratio as *TF* to *FQ*;

therefore as *ES* is to *DI*, so is *TF* to *FQ*.

And as *TF* is to *FQ*, so is *JH* to *HQ*;

and *JH* is equal to *TR*,

therefore as *ES* is to *DI*, so is *TR* to *QH*.

And as *GE* is to *ES*, so is *CT* to *TR*, because the two triangles [*GES* and *CTR*] are similar,

therefore as *EG* is to *DI*, so is *TC* to *QH*.

And *ID* is to *DG* as *HQ* is to *QC*,

therefore as *EG* is to *GD*, so is *TC* to *CQ*.

And angles *EGD*, *TCQ* are equal, and therefore the two triangles are similar,

therefore angles *GDZ*, *CQF* are equal.

And angles *DGZ*, *QCF* are equal,
therefore as *DZ* is to *ZG*, so is *QF* to *FC*.
And *QF* is equal to *FC*,
therefore *ZD* is equal to *ZG*.
And that is what we wished to prove.

[Lemma VI: Figure 8]

Again, let the right-angled triangle *ABG* have the angle *B* right; let *AB* be produced on the side of *B*, and let point *D* be given on *BG*; and, further, let *E* to *Z* be a given ratio; and we wish to draw from *D* a line,
such as line *TDK*, so that
as *TK* is to *KG*, so is *E* to *Z*.

Join *AD*,
and let *AD* to *H* be as *E* is to *Z*;
draw *DM* parallel to *BA*, so that angle *MDG* will be right;
on the triangle *MDG* describe a circle with diameter *MG*;
construct angle *DMC* equal to angle *DAG*;
from point *C* draw *CLN*, so that line *LN* will be equal to line *H*;
join *DKN* and carry it through on the side of *D* to *T*;
and join *GN*.

Therefore angle *DNG* will be equal to angle *DMG* which is equal to angle *BAG*.
But angle *NKG* is equal to angle *AKT*,
therefore line *KT* will meet line *AB*, say at point *T*,
and, therefore, triangles *ATK*, *NGK* will be similar;
therefore as *TK* is to *KG*, so is *AK* to *KN*.

And angle *DNC* is equal to angle *DMC* which is equal to angle *DAG*,
therefore triangles *AKD*, *NKL* are similar;
therefore as *AK* is to *KN*, so is *AD* to *NL*, which is the same as *E* is to *Z*;
therefore as *TK* is to *KG*, so is *E* to *Z*.
And that is what we wished to prove.

And it was shown earlier that there issue from point *C* two lines such that the segment of each of them that lies between the circle and the diameter will be equal to the given line. Thus if two such lines are drawn from *C*, then there will issue from point *D* two lines in the given ratio; but the two angles produced at point *G* will be unequal, I mean angle *TGK* and the angle corresponding to it.

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