Al-Kāshī's Miftāḥ Al-Ḥisab, Volume I Arithmetic: Translation and Commentary

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Contents

1 Preface

It is unfortunate that one of the most important mathematics books of the medieval Islamic civilization,[1] namely *Miftāḥ al-Ḥisab* written by *al-Kāshī* in 1427, has not been fully translated to English before. In fact, until the middle of the twentieth century it was unknown to modern researchers [27]. According to Rashed [31], traditional history of mathematics was shaken by Luckey's discovery of this monumental work. *Miftāḥ Al-Ḥisab* is written in Arabic and has never been fully translated to another language except for Russian [33]. Also, two small sections of *Miftāḥ* are translated to English, one on root extraction [16], another one on measuring the areas of muqarnas [18]. Before *Miftāḥ* was discovered by modern researchers, the discovery of decimal fractions was incorrectly attributed to Simon Stevin [29]. *Miftāḥ* contains a systematic treatment of decimal fractions, yet al-Kāshī is not the inventor of decimal fractions. According to best available sources, decimal fractions were first introduced in the middle of the tenth century by Abū al-Ḥasan Aḥmad ibn Ibrāhīm al-Uqlīdisī [36].[2] The story of *Miftāḥ* and the discovery of the contributions of al-Kāshī to mathematics and astronomy by modern researchers is a reflection of a larger story in the history of science that is called "the classical narrative" by George Saliba [37]. One of the tenets of this narrative is the assumption that the scientific progress and production in the Islamic World stopped well before the 15th century. By presenting this book, we hope to contribute to the growing body of evidence that quells this narrative. For a long time, a Euro-centric version of history of science has been in the making with a huge deficit in research on primary sources of scientific works in the Islamic civilization as well as other civilizations such as Chinese and Indian. Since the second half of the 20th century much research has been conducted to remedy this situation and a lot of progress has been made. However, more work is needed on two fronts. i) There is still a wealth of primary sources waiting to be studied in many parts of the world, and ii) more efforts is needed to disseminate accurate information about Islamic science to counter and repair the damaging effects of the classical narrative. To illustrate, the personal stories of the authors of this book are really instructive. The first author was born and raised in the capital of the Ottoman Empire,[3] have been unaware of the profound contributions of Islamic scholars to mathematics and science until a few years ago. It was disappointing to learn that the unique copy of al-Uqlīdisī's *Kitāb al-fuṣūl fī al-Ḥisāb al-Hindī* was in Yeni Cami Library in his hometown, yet nobody studied it until 1960's [35]. When it was finally translated to English and made widely available [36], incorrect attributions about the origin of decimal fractions had been, and are still circulating. The second author was born and raised not far from where the famous mathematician Leonardo of Pisa, otherwise known as Fibonacci, have been studying in Algeria. The contributions of Fibonacci are well documented, yet nothing transpires about his teachers in Algeria. Therefore, studying primary sources, making them available to a larger audience, and disseminating research findings to

[1]We will be using the terms "Islamic Civilization" and "medieval Islamic Civilization" in a very broad sense. We are particularly referring to the medieval Islamic civilization for which the time period extends approximately from late 7th century to the 16th century (inclusive). Geographically, it spans a large region –from Spain to the west to China, and India to the east. Therefore, it encompasses much diversity in terms of languages, ethnicity, and cultures. It also contains many different political powers and organizations (such as Umayyads, Abbasids, Fatimids, al-Andalus, Seljuks, Ottomans, Safavids, and others). It was ethnically and religiously highly diverse in which individuals from many different backgrounds contributed to scientific knowledge and progress, being a Muslim was not a prerequisite for this contribution. Arabic was the language of science during this time period, and it has been the language of religious studies. Once again, we use the term in a very broad sense and in no way do we imply a monolithic culture or civilization.

[2]It is astonishing that false information about the invention of the decimal fractions is still repeated in books published in the 21st century (e.g. [17]) and when the authors discuss the history of decimal fractions they leave out the most relevant figures such as al-Uqlīdisī and al-Kāshī.

[3]That is Istanbul in modern day Turkey.

the general public are very important activities. The first author found manuscript copies of *Miftāḥ* in Süleymaniye library in Istanbul and realized that they seem to be largely unknown in the literature. We use some of those manuscripts in this translation. It is our belief that the full translation of *Miftāḥ* will contribute to the writing of a universal history of mathematics that is inclusive. We are confident that a careful reading of *Miftāḥ Al-Ḥisab* will provide us with a broader, more accurate overview of the state of knowledge of mathematics in the eastern part of the Muslim world in the fifteenth century. Further research that can be generated from this book will undoubtedly lead to a better understanding of mathematical concepts and techniques used at the time. We hope that publishing *Miftāḥ Al-Ḥisab* in three volumes, Arithmetic, Geometry, and Algebra will serve this purpose well.

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Both authors greatly appreciate and acknowledge the work of Ghada Bakbouk for her meticulous editing of the the translation of this volume and correcting many errors. Any remaining errors are due to the authors.

We dedicate this volume to the memory Professor Fuat Sezgin, a prominent researcher in the history of mathematics and science in the Islamic civilization who recently passed away. He was a remarkable and leading scholar in the field who was a source of great inspiration for us. His exceptional dedication and monumental work will continue to be important for future researchers. We humbly hope that this volume will contribute to the purpose to which he dedicated his long and productive life.

Figure 1: Iranian stamp featuring an image of al-Kāshī with an astrolabe in the background

2 Introduction

2.1 A Biography of al-Kāshī and a Brief History

On March 2, 1427 ([5], p. 32; [18]), Ghiyāth al-Dīn Jamshīd bin Mas'ūd bin Maḥmood al-Tabīb al-Kāshī (or al-Kāshānī) completed a monumental book in Arabic on arithmetic called *Miftāḥ al-Ḥisāb*. The most commonly used translation of this title, the one which we prefer as well, is "Key to Arithmetic". Other possible translation include "Calculator's Key" and "Key to Calculation". Henceforth we will be referring to it as *Miftāḥ* and its author as al-Kāshī. This encyclopedic work includes three general subjects: arithmetic, geometry, and algebra. It comprises five treatises on arithmetic of integers, arithmetic of fractions, arithmetic of sexagesimal numbers, geometry and measurement, and algebra. Al-Kāshī dedicated the *Miftāḥ* to Ulugh Beg who himself was a prominent scientist and was the ruler of Samarkand at the time. In Berggren's assessment *Miftāḥ* was "the crowning achievement of Islamic arithmetic, and truly a gift fit for a king" ([13], p. 22). Saidan agrees by stating "*Miftāḥ* represents the peak of Islamic arithmetic" ([36], p. 29).

We do not know all the details of al-Kāshī's life. He was born in Kāshān, in modern day Iran some 150 miles south of the capital city Tehran. Therefore, he is also (perhaps more accurately) known as Al-Kāshāni but al-Kāshī is more commonly found in the literature. He was born in the later part of the 14th century but we do not know for certain in what year. Some authors seem to think that he was born in 1380 (e.g. [42],[43],[29], [19]) but the source of this date is not clear. According to Suter, al-Kāshī died in the year 1436 [41] (repeated by other authors as well, e.g, [34]). However, Kennedy gives the date of June 22, 1429 (Ramadan 19, 832 A.H.) based on a note on a manuscript copy of one of al-Kāshī's works [44]. This is the commonly accepted date of al-Kāshī's death.

His given name is Jamshīd, his father's name is Mas'ūd and his grandfather's name is Maḥmood. He has several nicknames best known of which is al-Kāshī. Giyath Al-Din means the rescuer of the religion. Al-Tabīb means physician. This is due to the fact that he practiced medicine as a profession before devoting his time totally to the study of mathematics and astronomy later in his life. He is known as al-Kāshī (or Al-Kāshāni) because he is from Kāshān. Assuming a nickname based on the place of birth was common in the Arabic and Islamic culture. There are many other famous Islamic scholars who are best known by such nicknames such as al-Khwārizmī and al-Ṭūsī.

The first dated event that we know of about his life is the observation of a lunar eclipse in his hometown of Kāshān on June 2, 1406, a date recorded in his *Khaqani zij* as the first date of a series of lunar eclipses he observed. During al-Kāshī's early years, Timur (Tamerlaine, 1336-1405) was conquering vast regions including central Iran. It was a difficult time for the people of the region with widespread poverty and turmoil ([44], [18]). Apparently, al-Kāshī managed to study mathematics and astronomy during this time. The conditions improved after Shah Rukh Mirza (1377-1447) took over the reign upon his father's death. Al-Kāshī's best times came after the young prince Ulugh Beg (1394-1449) became the governor of Samarkand, a city in Transoxiana in modern day Uzbekistan, after Shah Rukh Mirza moved the capital to Herat in modern day Afghanistan. Prince Ulugh Beg was a scientist himself excelling in mathematics and astronomy who supported many scholars and students. He gathered some of the best scholars of his time in Samarkand, and established an observatory there which was one of the best observatories ever built until that time. Ulugh Beg also established a madrasa –a school for advance studies in theology and sciences– in Samarkand between 1417 and 1420. The construction of observatory began after the completion of the madrasa [24]. Therefore, Samarkand became a major center of research and learning of the time [5, 6].

We know that al-Kāshī joined the scientific circle of Ulugh Beg upon his invitation. We do not know exactly when al-Kāshī moved to Samarkand but it must be around 1420. Different dates are mentioned for this move by researchers such as 1417 in ([42], p. 6), 1418 in ([13], p. 21), and 1421 in ([9]). By the time he was invited to Samarkand, al-Kāshī must have proved his scientific ability. Indeed, we know that he completed several works between 1406 and the date of his move to Samarkand. For example, in 1407 he completed *Sullam Al-Sama -Ladder of the Heaven on resolution of difficulties met by predecessors in determination of distances and sizes*. In 1410-1411, he wrote *Mukhtasar dar ilm-i hay'at-Compendium of the science of astronomy* dedicated to Sultan Iskandar, one of the rulers of Timurid dynasty. In 1413-1414, al-Kāshī wrote *Khaqani zīj*[4] and dedicated it to Ulugh Beg. In the introduction of this book al-Kāshī complains about living in poverty while working on mathematics and astronomy, and he says he could not have completed this work without Ulugh Beg's support. Al-Kāshī sought to get patronage of a ruler and this might be his first success.

Al-Kāshī did some of his best work in Samarkand under the patronage of Ulugh Beg. This includes *Miftāḥ Al-Ḥisab*, an encyclopedic book on elementary mathematics, and his remarkable approximations to π and $\sin(1°)$. Two letters of al-Kāshī in Persian to his father provide us with interesting facts and insights into the scientific environment of Ulugh Beg's court and personalities of some of the important scholars there. From these letters and other sources we learn that Ulugh Beg tolerated al-Kāshī's lack of court etiquette thanks to his excellent command of mathematics and astronomy [25, 39, 9].

Al-Kāshī's letters were discovered in the second half of the 20th century at different times. The second letter was discovered first and published in 1960 independently by two different researchers. Kennedy gave an English translation of the letter together with a commentary in [25], and Sayili gave both a Turkish translation and an English translation in [39]. The first letter was discovered later by Bagheri who published it with an English translation [9]. Al-Kāshī explains that he wrote the second letter and repeated a lot of information in case the first letter, which was sent via the merchants of Qum, might have been lost. From the details given in the letters we can infer that al-Kāshī's father was a learned man who knew mathematics and astronomy. The main topics he discusses in his letters are his professional triumphs and how he distinguished himself from other scientists in the court. Moreover, he speaks highly of the character of his patron Ulugh Beg, and the progress in the construction of the observatory that was being built at the time of the writing when the letters. In describing his scientific achievements, al-Kāshī refers mostly to the scientific staff of Ulugh Beg in disdain thinking of their incompetence. Al-Kāshī has a very high opinion of Ulugh Beg himself, and only very few of other scientists in the group such as Qāḍīzāde al-Rūmī (1364-1436) who was a teacher of Ulugh Beg. The letters depict Ulugh Beg more as

[4] zīj is a book of astronomical tables

a scientist and scholar than as a ruler or statesman who attended many of the meetings in which scientific matters are discussed by the scholars of his court. We learn that Ulugh Beg visited Maraga observatory in his childhood which probably made an impression on him. We also learn that serious scientific studies at Samarkand had been undertaken for about twelve years, and that there were at least sixty to seventy mathematicians among Ulugh Beg's staff in addition to astronomers. Astronomy was one of the most important scientific disciplines in the Islamic tradition ([37]), and "the observatory, as the scientific institution we know today, was born and developed in the Islamic world" ([13], p. 21). The Samarkand observatory that was established by Ulugh Beg turned out to be one of the most important observatories in the Islamic world [38].

Not satisfied with zīj's compiled before his time, Ulugh Beg decided to conduct fresh observations and built the Samarkand observatory. It would be reasonable to infer that the building of the observatory was planned before al-Kāshī arrived at Samarkand and started shortly after his arrival. We learn from al-Kāshī's letters that Ulugh Beg followed al-Kāshī's advice on certain technical aspects of the building during the construction and changed the original plan upon being convinced by al-Kāshī's explanations. Al-Kāshī was the first director of the astronomical observations at the observatory [39] but unfortunately, he passed away before the full study was completed. Qāḍīzāde al-Rūmī took over but sadly, his life was not long enough either to complete the project either. Ali Qushji (1403-1474) completed the project. Al-Kāshī's letters are among the best sources of information on the instruments of the Samarkand observatory.

Al-Kāshī praises Ulugh Beg in these letters in several ways. For example, he mentions that Ulugh Beg is well versed in both religious sciences and mathematics. (Translation from [39])

> Truth is that, first of all, he knows most of the holy Quran by heart, and he has a ready knowledge of its exegeses. For each occasion he cites an appropriate verse of the Quran, and he makes elegant quotations. Every day he reads fluently and in the proper manner two sections from the sacred book in the presence of experts who know the whole of the Quran by memory, and no mistakes occur. His knowledge of grammar and syntax is very good, and he writes Arabic extremely well. Likewise, he is well versed in jurisprudence, and he is acquainted with logic and the theory of literary style, as well as with the principles of prosody.
>
> His majesty has great skill in the branches of mathematics. His accomplishment in these matters reached such a degree that one day, while riding, he wished to find out to what day of the solar year a certain date would correspond which was known to be a Monday of the month of Rajab in the year 818 and falling between the 10th and the 15th of the month. On the basis of these data he derived the longitude of the sun to a fraction of 2 minutes by mental calculation while riding on horseback, and when he got down he asked this servant to check his result.
>
> It is true that, as in mental calculation it is necessary to retain quantities in one's mind and to derive others from them, and because in the faculty of memory there is a shortcoming, he [i.e. al-Kāshī himself] could not find the result (correctly) in degrees and minutes and was content with degrees only. But it is not given to any person of our time to do the like; no one else is capable of it.

He also says Ulugh Beg is very generous and supports a large number of students seeking knowledge. (Translation from [9])

> His royal majesty [Ulugh Beg] had donated a charitable gift amounting to 30,000 kopaki dinars, of which 10,000 had been ordered to be given to students. [The names

Figure 2: Ulugh Beg's Statue in Samarkand, Uzbekistan. (Image source: wikipedia)

> of the recipients] were written down: [thus] 10,000-odd students steadily engaged in learning and teaching, and qualifying for a financial aid, were listed.

These numbers show that Samarkand was a major center of learning at the time. This is supported by further facts mentioned in al-Kāshī's letters. He states that in addition to the students studying with financial aid, there are about 500 persons among notables and their sons who began studying mathematics at twelve places. He says "there are 24 calculators some of whom are also astronomers and some began [studying] Euclid ['s *Elements*]" [9]. He compares the situation with Kāshān where one or two persons may be associated with a given discipline, and concludes that it is not a good environment for healthy discussion of scientific matters.

Ulugh Beg is also described, in these letters, as a kind and open-minded person who is keen on rigorous investigation in science. He listens to all points of views, lets everyone express their opinions and make their case, gets engaged in arguments with students and experts. He lets discussions go on until the issues become clear to everyone. He does not approve submission to the authority without convincing proofs [39].

> He is indeed good-natured to the utmost degree of kindness and charity, so that, at times, there goes on, at the madrasa, between His Majesty and the students of the seeker of knowledge so much arguing back and forth on problems pertaining to any of the sciences that it would be difficult to describe it. He has ordered, in fact, that this should be the procedure, and he has allowed that in scientific questions there should be no agreeing until the matter is thoroughly understood and that people should not pretend to understand in order to be pleasing. Occasionally, when someone assented to His Majesty's view out of submission to his authority, His Majesty reprimanded him by saying "you are imputing ignorance on me." He also poses a false question, so that if anyone accepts it out of politeness he will reintroduce the matter and put the man to shame.

A large part of al-Kāshī's letters is devoted to communicate to his father that he is indeed the best, most proficient, and competent scientist in the group. He makes this point in several ways and gives many examples. He narrates several occasions where other scientists spent a lot of time on a given problem but they could not solve it, yet when al-Kāshī enters the picture he solves the problems quickly and easily. In some cases, other scholars think that there is a missing information in the problem statement but al-Kāshī shows that this is not the case and solves the problem. Other times, he publicly challenges other scholars on scientific issues and proves his points. He says this made him famous. He also writes about strong words of praise for himself by Ulugh Beg about his character and scientific competency [39].

> As to the complimentary remarks of His Majesty, of which mention has been made above, the situation is that no week passes without some friends reporting to this servant that His Majesty made such and such remarks tonight or today concerning me. They are as follows: "He has the knowledge of things ready at hand"; "He knows extremely well"; "His knowledge is superior to that of others"; "His knowledge is more readily at his disposal and more substantial than is the case with Qāḍīzāde"; 'His mind works better in this science than that of Qāḍīzāde "; "Mawlana Giyāth al-Dīn knows all the parts of this science, and he solves at once or in a single day a difficulty which takes Qāḍīzāde ten days to disentangle."
>
> He said, likewise, "He is a good and kind-hearted man. All those who have access to our circle, whether they be of the notables or not, have not restrained themselves but have quarreled with people and have transgressed their limits although I have shown them little courtesy. On the other hand, although I have extended much courtesy and many grants to Mawlana Giyath Al-Din and even though he is always honored with access to our company, he has not had any quarrels with anybody and nor has anyone complained about him." "He has not, out of greed, resorted to speak and gossip behind people's backs"; "He conducts himself very well." He has said things of this nature many times.

Al-Kāshī thinks that Qāḍīzāde the most knowledgeable scholar in the group but he narrates situations that show he is not as good as al-Kāshī. An immediate logical consequence of this is that al-Kāshī is the very best scholar in the entire group of Ulug Beg's scientific staff. He tells his father that he followed his advice to focus on only one subject (in this case it is astronomy) and he understands the reason behind this advice as "because occupation with another subject would indeed distract me from astronomical observations; second, because of my occupation with another art in which I may be a beginner, there may occur in my discussions or compositions some defect or error which people would bring to bear on the other arts [in which I am adept]" [9]. Evidently, another purpose of al-Kāshī's letters to his father was to refute some rumors his father heard through a person named Shams Al-Din. He makes it clear that Shams Al-Din's statements about him are in fact false.

Al-Kāshī had a productive time in Samarkand but unfortunately it did not last long. He died on June 22, 1429 at the observatory in which he was deeply involved. Ulugh Beg says the following for al-Kāshī in the preface of his own zīj' written some years after al-Kāshī's death: "the remarkable scientist, one of the most famous in the world, who had a perfect command of the sciences of the ancients, who contributed to its development, and who could solve the most difficult problems" [44].

Figure 3: Ulugh Beg's Observatory in Samarkand, Uzbekistan. (Image from Wikimedia Commons)

2.2 List of al-Kāshī's Known Works

In addition to *Miftāḥ* which is the most comprehensive work of al-Kāshī, we know of the following works of al-Kāshī. Al-Kāshī's works were collected in *Majmu'*(*Collection*), Tehran 1888. The following list of al-Kāshī's works can be found in [5] and [44].

- **Zij Khaqani**
  Zīj' is a Persian word used for astronomy books containing tables that are important for astronomical calculations. This work was written around 1413-1414 before al-Kāshī moved to Samarkand. Written in Persian, it was an update and completion of the *Zij Ilkhani* compiled by famous Nasir al-Din Al-Tusi about 150 years earlier. Bartold believed that al-Kāshī dedicated this work to Shah Rukh who was a patron of science in Herat [10]. However, Kennedy established that the zij was dedicated to Ulugh Beg, son of Shah Rukh and ruler of Samarkand [44]. This may be al-Kāshī's first attempt to secure funding under Ulugh Beg's patronage. Al-Kāshī used iterative methods to obtain approximate value of the third of each arc [5]. This zij contains six treatises and the tables in it are in sexagesimal system [44]. Al-Kāshī also states that he collected data from the works of earlier astronomers (astrologers) that did not show up in other tables, along with geometric proofs [5]. Arabic manuscripts are available in London, Oxford, and Istanbul [44].

- **The Treatise on Circumference**
  One of al-Kāshī's most remarkable achievements is his approximation of π accurate to 9 sexagesimal places, or 16 decimal places. According to Hogendijk [20], this is one of the highlights of the mediveal Islamic mathematical tradition. Before al-Kāshī, the previous record was 6 decimal place approximation by Chinese mathematicians. It took nearly two centuries before the Dutch mathematician Ludolf Van Ceulen surpassed al-Kāshī's accuracy with 20 decimal places. Written in 1424, the original title of this work is *Al-risala al-muhitiyya*. His motivation to compute π so accurately was to calculate the circumference of the universe so that the round-off error in the approximation would be no more than the width of a horse's hair. He set the accuracy of his approximation in advance and he used a polygon with $3 \cdot 2^{28} = 805,306,368$ sides. Hogendijk states that since al-Kāshī's text was not available in English translation until recently, incorrect or confused statements on the history of π often appeared in the western literature [20].

For example, al-Kāshī was not mentioned in *A History of Pi* [20], a popular book on the subject. Arabic manuscripts are available in Istanbul, Tehran and Meshed [44].

- **The Treatise on Chord and Sine**
Given al-Kāshī's dissatisfaction with earlier zijs, he calculated accurate values for sines and chords in this work to help with astronomical tables. He also calculated $\sin(1^\circ)$ to the same degree of accuracy of his computation of π. Unfortunately, the original text of this work is lost, however, Qāḍīzāde al-Rūmī's account of this treatise is available in the national library of Iran [8] and is translated to Russian as well. The outline of al-Kāshī's main method in this approximation is as follows. He first used the trig identity $\sin(3\theta) = 3\sin(\theta) - 4\sin^3(\theta)$ which he transformed to a cubic equation of the form $x = a + bx^3$. He then used an iterative method to obtain successive approximations to $\sin(1^\circ)$ that get more accurate at each step. His beautiful approach is still important in modern mathematics and is usually covered under the name of "fixed point iteration" in numerical analysis courses. See [1] for more details on the computation of $\sin(1^\circ)$. There is an edition of this manuscript in *Majmu'* [44].

- **The Zīj at-Tashilat (The Zīj of Simplifications)**
Al-Kāshī refers to this work in *Miftāḥ* among his works but it is non-extant. It probably included a simplified method of computing the positions of celestial bodies [24].

- **Risala dar Sharh-i Alat-i Rasad (Treatise on the Explanation of Observational Instruments)**
Written for Sultan Iskandar (Kara Yusuf of Black Sheep Turks (Karakoyunlu) dynasty) in 1416, al-Kāshī gives brief yet accurate descriptions of the constructions of eight astronomical instruments. A manuscript copy of this work is available in Leiden, which, according to F. F. Bartold, was written by al-Kāshī himself [5]. Moreover, Persian manuscripts are available in Leiden and Tehran[44].

- **Nuzha Al-Hadaiq (Delight of Gardens)**
Al-Kāshī wrote this treatise in 1416 (February 10, 1416) in Kāshān to which he made additions in Samarkand in 1426. A revised Persian version was written by an anonymous astronomer in Istanbul around 1490. [21]. It gives an explanation of how to build an instrument invented by al-Kāshī which he called "The Plate of Conjunctions" [5]. According to al-Kāshī, it is an instrument from which one can retrieve the calendars of planets, their widths, dimensions, distances from earth, eclipses and other related matters. One can describe this instrument, which has a similar shape as the astrolabe, as the diagram for the approximate graphical solutions of many problems related to the movement of the stars based on the mean values of their coordinates. Al-Kāshī's later additions include ten appendices which describe additional techniques to utilize the instrument [24]. Arabic manuscripts are available in London, Dublin and Bombay [44].

- **Sullam Al-Sama (The Ladder of the Sky or The Stairway of Heaven)**
al-Kāshī completed this work in his hometown in 1407. It is the earliest known work of al-Kāshī.The full title of this text in the area of astronomy is "on Resolution of Difficulties Met by Predecessors in Determination of Distances and Sizes (of Heavenly Bodies)" [29]. Al-Kāshī mentions this treatise in the preface of *Miftāḥ*. He implies that scholars before him had difficulties and disagreements about distances and sizes of the heavenly bodies. So, he decided to write this book in order to help future scholars. Arabic manuscripts are available in London, Oxford, and Istanbul [44]. There is an Iranian TV series with the same title, the Ladder of the Sky, about al-Kāshī's life which

was broadcast during the month of Ramadan of 2009. Full episodes are available at http://www.shiasource.com/ladder-of-the-sky

- **Talkith Al-Miftāḥ (Abridged Key [to Arithmetic])**
  Written before the *Miftāḥ* itself (in 1421), it was an early and abbreviated version which contains three treatises and about one eighth of the material in *Miftāḥ* [8]. Qurbani provided Persian translations of the chapters of *Talkhith* [30]. Arabic manuscripts are available in London, Tashkent, Istanbul, Baghdad, Mosul, Tehran, Tabriz, and Patna [44].

- **Miftāḥ al Asbab-fi al-Ilm al-Zīj (The Key of Reason in the Science of Astronomical Tables)**
  There is an Arabic manuscript in Mosul (120/306) [44], [34].

- **Risala dar Sakht-i Asturlab (The Treatise on the Construction of an Astrolabe)**
  There is a Persian manuscript in Mosul [44].

- **Ta'rib Al-Zīj (Arabization of the Zīj)**
  This is an Arabic translation of the introduction of Ulugh Beg's zīj which was in Persian. Manuscripts are available in Leiden and Tashkent [44].

- **Ilḥāqāt an-Nuzha (Supplement to the Excursion, 1427)**
  There is an edition of a manuscript in *Majmu'* [44].

- **Wujūh al-Amal al-Darb fī'l-Takht wa'l-Turāb (Ways of Multiplying Using a Dust Board)**
  There is an edition of an Arabic manuscript in *Majmu'* [44].

- **Natāij al-Ḥaqāiq (Consequences of Truths)**
  There is an edition of an Arabic manuscript in *Majmu'* [44].

- **Risāla fī Ma'rifa Samt al-Qibla min Dāira Hindiyya Ma'rufa (Treatise on the Determination of Direction of the Qibla from a Circle Known as Indian Circle)**
  There is an edition of an Arabic manuscript in Meshed (84) [44], [34].

- **Tuḥfat al Sultān fī Asbab al-Irfān (Gift to Sultan on Causes of the Science)**
  Oxford (1514). Treatise on astronomy dedicated to Amir-zade Ibrāhīm Sultan, son of Shahruh, [34].

- **Risala dar Hay'at (Treatise on Astronomy)**
  P-London (Sup. 27261), Yazd (Jami 439/5), [34].

- **Mukhtasar dar Ilm-i Hay'at (Concise Treatise on the Science of Astronomy)**
  P-London (869b), Treatise dedicated to Amir-zade Sultan Iskandar Bahadur, [34].

- **Risālat ḥall ashkāl Utārid (Treatise on Solution of Propositions on Mercury)**
  Meshed (5527), [34].

- **Risala-yi Kamaliyya (Treatise for Kamal al-Din)**
  P-Hyderabad (riyada 125-126), Astronomical treatise in 7 books and a conclusion written in Kashan in 1406, dedicated to vizier Kamal al-Din Mahmud. [34].

- **Risāla fī Tashīh Awsat al-Qamar min al-Arsad al-Khusufiyya (Treatise on Closer Definition of the Center of the Moon from Observations of Eclipses)** Cairo (riyada 898/23), [34].
- **Zīj al-Kāshī**-Meshed (5321) [34]
- **Notes on Linear Interpolation** -Cairo (Zaki 917/14) [34]

2.3 Manuscript Copies of *Miftāḥ*

Two different typeset and printed books of *Miftāḥ* that include the original Arabic manuscript together with some commentary and explanations are available. The first one is by Ahmad Sa'id Al-Dimirdash and Muhammad Hamdi Al-Hifni al-Shaykh published in Cairo in 1969 [6] in which commentaries are in Arabic. The other one was published in 1977 in Damascus by Nabulsi [5]. Nabulsi edition has most commentaries in Arabic and some in French as well. It lists the following known manuscript copies of *Miftāḥ* at the time of its publication.

1. Orginal book of Miftāḥ Al-Ḥisab, written by Jamshīd bin Mas'ud bin Mahmoud al-Kāshī March 3rd, 1427 (830 h.), but it is missing.
2. Al Burgandy's manuscript, which is also missing. According to Muhammad Al-Sadiq Al-Arassengi in the Zahiri manuscript, it is written by Abdul Ali Al-Burgandy on Tuesday, 17th day of Dhu Al-Hijja in 889 H., which corresponds to January 5, 1485.
3. The Leiden manuscript, written in 1558 (965 h).
4. The British Museum manuscript, written in 1589 (887 h).
5. The Zahiri manuscript written in 1691 (1102 h).
6. The Leningrad manuscript, written in 1789 (1204 h).
7. The Scientific Library of Prussia manuscript in Berlin, written in 1886 (1303-1304h).
8. The Public Scientific Library in Berlin manuscript (spr 1824)
9. The Berlin Institute for History of Medicine and Science, number 1 and 2.
10. The National Library of Paris, number 5020.

Dimirdash and Muhammad Hamdi edition also mentions the manuscripts 3,4,6-10. They state that they relied on Leiden manuscript [6]. Their list does not include the Zahiri manuscript but includes one that does not apppear in the list above, namely, a stone-print copy in Tehran, located in the Tayomrian cabinet number 255-Math. Nabulsi says he used the following manuscripts: Zahiri, London, and Leiden as well as the print book [6] and the Russian translation by Rosenfeld and Youschkevitch published 1956 [33]. The Zahiri manuscript is the main source of his investigation. Nabulsi regards Zahiri manuscript as the oldest known one, gives the following information about it ([5], p. 31) and includes a picture of the first and last pages.

> The Zahiri manuscript, number 7795 amongst the collection of books in the Zahiri library in Damascus. It is written on 128 sheets of paper of dimensions 21x12 cm, the handwriting is nice, the titles are written in red, has commentaries, it has eroded from the sides, embroidered with gold (written by Abd Al Ali bin Muhammad Al Burgandi in 889 then copied by Ibn Muhammad Jafar Sadiq Al Arasinji in 1103).

Nabulsi notices that time intervals between the original work and its subsequent manuscripts listed above are 73 years, 31 years, 102 years, 98 years and 97 years. He concludes that there must be several other manuscripts of *Miftāḥ* that are still missing to this day. The manuscript we are using is not in Nabulsi's list, and it appears to be one of the manuscripts he suspected was missing. This manuscript is in Süleymaniye library in Istanbul in its Nuruosmaniye branch with new record number 2532 and old record number 2967. It was written in Ramadan 854 H., which corresponds to October 1450. There are several other manuscripts of *Miftāḥ* in Süleymaniye library which are labeled as Atıfefendi1719, Esadefendi13175, Fatih5421, Hamidiye883, and Hüsnüpaşa1268. Some of them appear to be incomplete and some of them do not seem to contain a date of completion. The manuscript that we are using, Nuruosmaniye2967, is the oldest one in the collection, and it is complete. Based on available data in the literature, Nuruosmaniye2967 manuscript seems to be one of the oldest, possibly the oldest, available manuscript copy of *Miftāḥ* to this date. Note that the Zahiri manuscript that Nabulsi considers "the oldest known manuscript of *Miftāḥ* so far" was originally written in 889 H. (1484 CE) which is later than the date of Nuruosmaniye manuscript we use (854H-1450 CE). The only reference in the literature to the Nuruosmaniye manuscript that we are aware of is in [36]. The translators of section 5 of treatise 3 of Miftāḥ(on root extraction in sexagesimal system) used the following three manuscript copies.

- Princeton University MS ELS 1189
- India Office Arabic MS L756
- University of Leiden MS Or. 185.

2.4 Modern Researchers' Assessment of Miftāḥand al-Kāshī's Work

We include here a few selected quotes from modern researchers about the significance of al-Kāshī's work in general, and *Miftāḥ Al-Ḥisab* in particular.

- The achievements of al-Kāshī represent the culmination of Islamic civilization's progress in mathematics. Some of these achievements reappeared or rediscovered later in Europe. (F. Sezgin, [40], p. 66)

- Key to Arithmetic is the crowning achievement of Islamic arithmetic. (J. J. Berggren, [13], p. 22)

- Miftāḥrepresents the peak of Islamic arithmetic. (A. S. Saidan, [36], p. 29)

- It was only 1948 that Luckey presented the first extensive and detailed study of al-Kāshī's work, and the traditional historical picture was explicitly shaken. (R. Rashed, [31], p. 149).

- In the richness of its contents and in the application of arithmetical and algebraic methods to the solution of various problems, including several geometric ones, and in the clarity and elegance of exposition, this voluminous textbook [that is, *Miftāḥ*] is one of the best in the whole of medieval literature; it attests to both the author's erudition and his pedagogical ability. Because of its high quality the Miftāḥwas often recopied and served as a manual for hundreds of years. (A. P. Youschkevitch and B. A. Rosenfeld, [44], p. 256)

- It is in the work of Giyath Al-Din Jamshīd al-Kāshī in the early fifteenth century that we first see both a total command of the idea of decimal fractions and a convenient notation for them ... (V. J. Katz, [23], p. 270)

- The power and elegance of computational algorisms developed by Islamic mathematicians of the ninth through the fifteenth centuries has only recently come to be appreciated. The most sucessful of these computers was the Iranian scientist, Jamshīd Ghiyāth Al-Din al-Kāshī (E. S. Kennedy, [26], p. 522)

- Al-Kāshī was first and foremost a master computer of extraordinary ability, witness his facile use of pure sexagesimals, his wide application of iterative algorisms, and his sure touch in so laying out a computation that he controlled the maximum error and maintained a check at all stages. (E. S. Kennedy, [24], p. 8)

- In the determination of π, and in computational mathematics as a whole, al-Kāshī was a pioneer. (J. P. Hogendijk, [20], p. 85)

- Key to Arithmetic was used for centuries by astronomers, architects, artisans, surveyors, and merchants as a textbook in the Islamic world (J. Freely, [15])

- Al-Kāshī's most impressive mathematical work was *Key to Arithmetic*. The work is a major text intended to be used in teaching students in Samarkand, in particular al-Kāshī tries to give the necessary mathematics for those studying astronomy, surveying, architecture, accounting and trading. (J. J. O'Connor and E. F. Robertson, [29])

- The observatory, as the scientific institution we know today, was born and developed in the Islamic world (J. L. Berggren, [13], p. 21))

- From the standpoint of longevity and work therefore the Samarqand Observatory was one of the most important observatories of Islam, and it probably was the most important... Giyāth al-Din Jamshīd and Muin Al-Din-i Kāshī prepared the plan for the observatory... Giyath Al-Din was the first director of the Samarqand observatory ... (A. Sayili, [38], p. 265, 266, 271.)

- There is little doubt that al-Kāshī was the leading astronomer and mathematician at Samarkand and he was called the second Ptolemy by an historian writing later in the same century. (J. J. O'Connor and E. F. Robertson, [29])

- Fulfilling the needs of accountants, engineers, mathematicians, surveyors, lawyers, and others, this book is unprecedented amongst mathematical works in the middle ages in terms of its perfection, organization, and clarity of explanation (N. Nabulsi, [5], p. 28)

- It [root extraction algorithm of al-Kāshī in *Miftāḥ*] is an example of the powerful and sophisticated numerical methods developed in the Muslim East, methods yielding results of unprecedented precision for their time, and which reached their culmination in the work of al-Kāshī(A. Dakhel, [16], preface, v viii)

- Not only was this [approximation of $\sin(1^\circ)$] the most fascinating and creative method of approximation, but it was the first approximation method in the history of mathematics, and the most significant achivement in medieval algebra (M. K. Azarian, [8], p. 39)

- In the Ottoman Empire mathematicians called any excellent mathematician by the name al-Kāshī (O. Taani, [42], p. 4 quoting from [22] p. 236)

- My dissertation demonstrates that there is a great need to have a complete translation of *Miftāḥ* to English, and a more detailed description of al-Kāshī's mathematics and methods. (O. Taani, [42], p. 4)

2.5 Pedagogical Aspects of Miftāḥ

It is clear that *Miftāḥ* was written as a textbook and a practical guide for people in various professions that need arithmetic and elementary mathematics as mentioned by al-Kāshī himself in the introduction.

> I solved many problems I was asked by expert mathematicians either for testing me or for their own learning. Some of those problems could not be solved by one of the six algebraic forms. Through these works I acquired a lot of knowledge to solve elementary mathematical problems in the easiest, most beneficial and most efficient ways using clear methods. I wanted to clarify and compile them so that it becomes a guide to anyone interested. Therefore, I wrote this book and collected all that professional calculators need, avoiding tiring length and annoying brevity. I presented rules for most operations in tables to make them accessible for engineers.

It is also clear that *Miftāḥ* is not a theoretical book as the author did not make any effort to prove or justify the algorithms or procedures even by the standards of that time. The title suggests that arithmetic is viewed as the key in solving any problem that requires calculations. In the introduction of *Miftāḥ*, al-Kāshī describes arithmetic as follows: "Arithmetic is the science of rules to determine numerical unknowns from specific known quantities". Al-Kāshī also makes his purpose of making it a practical guide for anyone who needs arithmetic and elementary mathematics clear in the introduction:

Taani studied the content and pedagogy of *Miftāḥ* [42, 43]. His main findings about al-Kāshī's pedagogy in *Miftāḥ* are what he calls "multiple paths to practice mathematics" and his "exhaustive exclusive classification methods". Taani's formulation of "multiple paths" include the following features: multiple definitions, multiple algorithms, multiple formulas, and multiple solutions [43]. He explains that what inspired him to consider investigating multiple paths in al-Kāshī's pedagogy was student comments. He initially used some excerpts from al-Kāshī in his pre-calculus class and his goal was to "get data from students about using historical sources in the classroom". When he administered a questionnaire about the lesson, student comments on al-Kāshī's method of using multiple methods/approaches to solve a given problem captured his attention and that became a major topic in his dissertation.

As an example of multiple definitions of mathematical concepts, consider these two definitions of division by al-Kāshī in *Miftāḥ*

> i) In integers, it is dividing of the dividend in units of the divisor into a number of equal parts so that each share from the divisor is fixed. Such a share is called the quotient.

> ii) Its general definition is to obtain the number whose ratio to one is the same as the ratio of the dividend to the divisor.

Al-Kāshī often presents multiple methods to perform mathematical calculations. For example, he presents five different ways of performing multiplication of integers. He gives three different formulas to compute the area of a generic triangle, and four different formulas specifically for computing the area of an equilateral triangle. The last treatise of *Miftāḥ* contains a number of word problems and problems in inheritance (determining share of each heir according to Islamic inheritance laws). He shows multiple ways of solving these problems.

Taani suggests that his work may be used in a classroom in two different ways [43]. The first is directly using the primary text of al-Kāshī will let the students live the experience of discovering mathematics in the 15th century. The second is to follow al-Kāshī in presenting certain topics from multiple perspectives. He comments that the benefits of using multiple paths include: *providing flexibility of thinking*, *providing opportunities for comparison*, *creating a network of ideas*, and *improving creativity* [43].

Therefore, in addition to its profound significance in the history of mathematics due to its content, *Miftāḥ Al-Ḥisab* is also worthy of attention for its pedagogical aspects.

2.6 Possible Future Projects

As mentioned earlier, not only does al-Kāshī not give proofs or justifications for his algorithms and procedures, but he does not give any information about their origins. It is not clear what parts are his inventions or contributions and what parts are considered established results. A possible future work could be to research the history of the algorithms (such as finding square roots and higher degree roots) described in *Miftāḥ* to determine what exactly the contributions of al-Kāshī might be. On a different direction, it is quite likely that al-Kāshī's work may have influenced mathematicians who came after him. In a recent article [7] we pointed out one such possible connection. We observed that the famous Flemish mathematician Simon Stevin (1548-1620) presented root finding algorithms in his well known book *L'arithmétique"*, ("Arithmetic") published in 1585. On the surface, Stevin's algorithms look much different from al-Kāshī's. However, after careful examination, we observe that the underlying algorithm is the same, though there are some curious differences in details and implementation (see [7] for more info). To the best of our knowledge, no direct connection between the two mathematicians is known. Given al-Kāshī's influence on Ottoman mathematicians and the education system [22], one might imagine flow of ideas from Central Asia to Europe through the Ottoman land. This is definitely an area that requires further research.

2.7 Notes on Translation and the Purpose of This Work

This work is a translation with commentaries of the book *Miftāḥ Al-Ḥisab* written by al-Kāshī on March 2, 1427. We are using what we believe to be the oldest manuscript so far at hand, namely the Nuruosmaniye2967 manuscript which was originally written in (854H-1450 CE). Believing that this book would be a reference to researchers and students of history of Islamic Mathematics alike, we tried to strike a balance between a literal translation and more modern, smooth English translation. We kept in mind that on the one hand we would like to preserve the authenticity of the manuscript with a faithful translation that conveys all possible meanings that al-Kāshī might had in mind and on the other hand provide a meaningful text to a modern reader. So, we resorted to footnotes and square brackets in the text to further explain some of the ambiguities and odd to read sentences. Any inserted text from the margins of the manuscript are included in square brackets with footnote assertion to that effect. Also, we underlined some terms to indicate they are further explained in the glossary. Since Arabic is a language written and read from right to left, some tables have been reverted to read from left to right, except when there are no ambiguities the tables are left as they are. For example, tables about addition or multiplication of numbers are written and operated in Arabic the same way as in English, are left as they are. However, tables and numbers with sexagesimal digits are reverted when degrees, minutes, and seconds are not explicitly mentioned since they are in reverse order in Arabic and English. When the degrees, minutes and seconds are explicitly written with no confusion we leave the numbers and tables as they are. Moreover, many of the numbers and tables in the manuscript have been checked against several other manuscripts for accuracy. We indicate that in footnotes. Still, we strongly recommend caution when dealing with complex computations as some differences and ambiguities might arise between different manuscripts or even in different places within the same manuscript as the hand written letters and numbers might lead to some confusions.

Our goal in this work is to present the raw material of *Miftāḥ Al-Ḥisab* with some directions that could possibly enrich further studies by students and researchers. We purposely refrained from making any judgments on the nature, origin or the originality of the subject matter found in the *Miftāḥ*. Some of his work was in fact original, others were known to previous authors centuries ago. We leave the critical in-depth historical study to the invested reader to interpret, position, and effectively study al-Kāshī's work in its own right and/or in a historical context.