ARTICLES

Ideas of Calculus in Islam and India

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Introduction

Isaac Newton created his version of the calculus during the years from about 1665 to 1670. One of Newton's central ideas was that of a power series, an idea he believed he had invented out of the analogy with the infinite decimal expansions of arithmetic [**9**, Vol. III, p. 33]. Newton, of course, was aware of earlier work done in solving the area problem, one of the central ideas of what was to be the calculus, and he knew well that the area under the curve $y = x^n$ between $x = 0$ and $x = b$ was given by $b^{n+1}/(n+1)$. (This rule had been developed by several mathematicians in the 1630s, including Bonaventura Cavalieri, Gilles Persone de Roberval, and Pierre de Fermat.) By developing power series to represent various functions, Newton was able to use this basic rule to find the areas under a wide variety of curves. Conversely, the use of the area formula enabled him to develop power series. For example, Newton developed the power series for $y = \arcsin x$, in effect by defining it in terms of an area and using the area formula. He then produced the power series for the sine by solving the equation $y = \arcsin x$ for $x = \sin y$ by inversion of the series. What Newton did not know, however, was that both the area formula—which he believed had been developed some 35 years earlier—and the power series for the sine had been *known for hundreds of years* elsewhere in the world. In particular, the area formula had been developed in Egypt around the year A.D. 1000 and the power series for the sine, as well as for the cosine and the arctangent, had been developed in India, probably in the fourteenth century. It is the development of these two ideas that will be discussed in this article.

Before going back to eleventh-century Egypt, however, we will first review the argument used both by Fermat and Roberval in working out their version of the area formula in 1636. In a letter to Fermat in October of that year, Roberval wrote that he had been able to find the area under curves of the form $y = x^k$ by using a formula—whose history in the Islamic world we will trace—for the sums of powers of the natural numbers: "The sum of the square numbers is always greater than the third part of the cube which has for its root the root of the greatest square, and the same sum of the squares with the greatest square removed is less than the third part of the same cube; the sum of the cubes is greater than the fourth part of the fourth power and with the greatest cube removed, less than the fourth part, etc." [**5**, p. 221]. In other words, finding the area of the desired region depends on the formula

$$\sum_{i=1}^{n-1} i^k < \frac{n^{k+1}}{k+1} < \sum_{i=1}^{n} i^k.$$

Fermat wrote back that he already knew this result and, like Roberval, had used it to determine the area under the graph of $y = x^k$ over the interval $[0, x_0]$. Both men saw that if the base interval was divided into n equal subintervals, each of length x_0/n, and if over each subinterval a rectangle whose height is the y-coordinate of the right endpoint was erected (see FIGURE 1), then the sum of the areas of these N circumscribed rectangles is

$$\frac{x_0^k}{n^k}\frac{x_0}{n} + \frac{(2x_0)^k}{n^k}\frac{x_0}{n} + \cdots + \frac{(nx_0)^k}{n^k}\frac{x_0}{n} = \frac{x_0^{k+1}}{n^{k+1}}(1^k + 2^k + \cdots + n^k).$$

Similarly, they could calculate the sum of the areas of the inscribed rectangles, those whose height is the y-coordinate of the left endpoint of the corresponding subinterval. In fact, if A is the area under the curve between 0 and x_0, then

$$\frac{x_0^{k+1}}{n^{k+1}}\left(1^k + 2^k + \cdots + (n-1)^k\right) < A < \frac{x_0^{k+1}}{n^{k+1}}(1^k + 2^k + \cdots + n^k).$$

The difference between the outer expressions of this inequality is simply the area of the rightmost circumscribed rectangle. Because x_0 and $y_0 = x_0^k$ are fixed, Fermat knew that the difference could be made less than any assigned value simply by taking n sufficiently large. It follows from the inequality cited by Roberval that both the area A and the value $x_0^{k+1}/(k+1) = x_0 y_0/(k+1)$ are squeezed between two values whose difference approaches 0. Thus Fermat and Roberval found that the desired area was $x_0 y_0/(k+1)$.

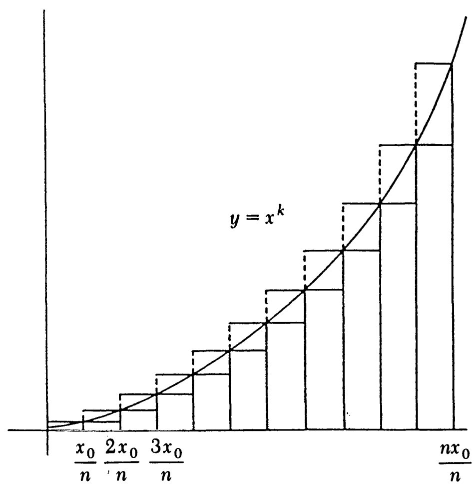

FIGURE 1

The obvious question is how either of these two men discovered formulas for the sums of powers. But at present, there is no answer to this question. There is nothing extant on this formula in the works of Roberval other than the letter cited, and all we have from Fermat on this topic, in letters to Marin Mersenne and Roberval, is a general statement in terms of triangular numbers, pyramidal numbers, and the other numbers that occur as columns of Pascal's triangle. (We note that Fermat's work was done some twenty years before Pascal published his material on the arithmetical triangle; the triangle had, however, been published in many versions in China, the Middle East, North Africa, and Europe over the previous 600 years. See [4], pp. 191–192; 241–242; 324–325.) Here is what Fermat says: "The last side multiplied by the next greater makes twice the triangle. The last side multiplied by the triangle of the next greater side makes three times the pyramid. The last side multiplied by the pyramid of the next greater side makes four times the triangulotriangle. And so on by the same progression in infinitum" [5, p. 230]. Fermat's statement can be written using the modern notation for binomial coefficients as

$$n\binom{n+k}{k} = (k+1)\binom{n+k}{k+1}.$$

We can derive from this formula for each k in turn, beginning with $k = 1$, an explicit formula for the sum of the kth powers by using the properties of the Pascal triangle. For example, if $k = 2$, we have

$$\begin{aligned} n\binom{n+2}{2} &= 3\binom{n+2}{3} = 3\sum_{j=2}^{n+1}\binom{j}{2} \\ &= 3\sum_{j=2}^{n+1}\frac{j(j-1)}{2} = 3\sum_{i=1}^{n}\frac{i(i+1)}{2} = 3\sum_{i=1}^{n}\frac{i^2+i}{2}. \end{aligned}$$

Therefore,

$$2\frac{n}{3}\frac{(n+2)(n+1)}{2} - \sum_{i=1}^{n} i = \sum_{i=1}^{n} i^2$$

and

$$\sum_{i=1}^{n} i^2 = \frac{n^3+3n^2+2n}{3} - \frac{n^2+n}{2} = \frac{2n^3+3n^2+n}{6} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}.$$

In general, the sum formula is of the form

$$\sum_{i=1}^{n} i^k = \frac{n^{k+1}}{k+1} + \frac{n^k}{2} + p(n),$$

where $p(n)$ is a polynomial in n of degree less than k, and Roberval's inequality can be proved for each k. We do not know if Fermat's derivation was like that above, however, because he only states a sum formula explicitly for the case $k = 4$ and gives no other indication of his procedure.

Sums of Integer Powers in Eleventh-Century Egypt

The formulas for the sums of the kth powers, however, at least through $k = 4$, as well as a version of Roberval's inequality, were developed some 650 years before the mid-seventeenth century by Abu Ali al-Hasan ibn al-Hasan ibn al-Haytham (965–1039), known in Europe as Alhazen. The formulas for the sums of the squares and cubes were stated even earlier. The one for squares was stated by Archimedes around 250 B.C. in connection with his quadrature of the parabola, while the one for cubes, although it was probably known to the Greeks, was first explicitly written down by Aryabhata in India around 500 [2, pp. 37–38]. The formula for the squares is not difficult to discover, and the one for cubes is virtually obvious, given some experimentation. By contrast, the formula for the sum of the fourth powers is not obvious. If one can discover a method for determining this formula, one can discover a method for determining the formula for the sum of any integral powers. Ibn al-Haytham showed in fact how to develop the formula for the kth powers from $k = 1$ to $k = 4$; all his proofs were similar in nature and easily generalizable to the discovery and proof of formulas for the sum of any given powers of the integers. That he did not state any such generalization is probably due to his needing only the formulas for the second and fourth powers to solve the problem in which he was interested: computing the volume of a certain paraboloid.

Before discussing ibn al-Haytham's work, it is good to briefly describe the world of Islamic science. (See [1] for more details.) During the ninth century, the Caliph al-Ma'mun established a research institute, the House of Wisdom, in Baghdad and invited scholars from all parts of the caliphate to participate in the development of a scientific tradition in Islam. These scientists included not only Moslem Arabs, but also Christians, Jews, and Zoroastrians, among others. Their goals were, first, to translate into Arabic the best mathematical and scientific works from Greece and India, and, second, by building on this base, to create new mathematical and scientific ideas. Although the House of Wisdom disappeared after about two centuries, many of the rulers of the Islamic states continued to support scientists in their quest for knowledge, because they felt that the research would be of value in practical applications.

Thus it was that ibn al-Haytham, born in Basra, now in Iraq, was called to Egypt by the Caliph al-Hakim to work on a Nile control project. Although the project never came to fruition, ibn al-Haytham did produce in Egypt his most important scientific work, the *Optics* in seven books. The *Optics* was translated into Latin in the early thirteenth century and was studied and commented on in Europe for several centuries thereafter. Ibn al-Haytham's fame as a mathematician from the medieval period to the present chiefly rests on his treatment of "Alhazen's problem," the problem of finding the point or points on some reflecting surface at which the light from one of two points outside that surface is reflected to the other. In the fifth book of the *Optics* he set out his solutions to this problem for a variety of surfaces, spherical, cylindrical, and conical, concave and convex. His results, based on six separately proved lemmas on geometrical constructions, show that he was in full command of both the elementary and advanced geometry of the Greeks.

The central idea in ibn al-Haytham's proof of the sum formulas was the derivation of the equation

$$(n+1)\sum_{i=1}^{n} i^k = \sum_{i=1}^{n} i^{k+1} + \sum_{p=1}^{n}\left(\sum_{i=1}^{p} i^k\right). \qquad (*)$$

Naturally, he did not state this result in general form. He only stated it for particular integers, namely $n = 4$ and $k = 1, 2, 3$, but his proof for each of those k is by induction on n and is immediately generalizable to any value of k. (See [7] for details.) We consider his proof for $k = 3$ and $n = 4$:

$$\begin{aligned}(4+1)(1^3+2^3+3^3+4^3) &= 4(1^3+2^3+3^3+4^3)+1^3+2^3+3^3+4^3\\ &= 4\cdot 4^3+4(1^3+2^3+3^3)+1^3+2^3+3^3+4^3\\ &= 4^4+(3+1)(1^3+2^3+3^3)+1^3+2^3+3^3+4^3.\end{aligned}$$

Because equation $(*)$ is assumed true for $n = 3$,

$$(3+1)(1^3+2^3+3^3) = 1^4+2^4+3^4+(1^3+2^3+3^3)+(1^3+2^3)+1^3.$$

Equation $(*)$ is therefore proved for $n = 4$. One can easily formulate ibn al-Haytham's argument in modern terminology to give a proof for any k by induction on n.

Ibn al-Haytham now uses his result to derive formulas for the sums of integral powers. First, he proves the sum formulas for squares and cubes:

$$\sum_{i=1}^{n} i^2 = \left(\frac{n}{3} + \frac{1}{3}\right)n\left(n + \frac{1}{2}\right) = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n}{4} + \frac{1}{4}\right)n(n+1)n = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}.$$

We will not deal with these proofs here, but only with the derivation of the analogous result for the fourth powers. Although ibn al-Haytham himself derives this result only for $n = 4$, he asserts it for arbitrary n. We will therefore use modern techniques modeled on ibn al-Haytham's method to derive it for that case. We begin by using the formulas for the sums of squares and cubes to rewrite equation $(*)$ in the form

$$\begin{aligned}(n+1)\sum_{i=1}^{n} i^3 &= \sum_{i=1}^{n} i^4 + \sum_{p=1}^{n}\left(\frac{p^4}{4} + \frac{p^3}{2} + \frac{p^2}{4}\right)\\ &= \sum_{i=1}^{n} i^4 + \frac{1}{4}\sum_{i=1}^{n} i^4 + \frac{1}{2}\sum_{i=1}^{n} i^3 + \frac{1}{4}\sum_{i=1}^{n} i^2.\end{aligned}$$

It then follows that

$$(n+1)\sum_{i=1}^{n} i^3 = \frac{5}{4}\sum_{i=1}^{n} i^4 + \frac{1}{2}\sum_{i=1}^{n} i^3 + \frac{1}{4}\sum_{i=1}^{n} i^2$$

$$\frac{5}{4}\sum_{i=1}^{n} i^4 = \left(n + 1 - \frac{1}{2}\right)\sum_{i=1}^{n} i^3 - \frac{1}{4}\sum_{i=1}^{n} i^2$$

$$\begin{aligned}\sum_{i=1}^{n} i^4 &= \frac{4}{5}\left(n + \frac{1}{2}\right)\sum_{i=1}^{n} i^3 - \frac{1}{5}\sum_{i=1}^{n} i^2\\ &= \frac{4}{5}\left(n + \frac{1}{2}\right)\left(\frac{n}{4} + \frac{1}{4}\right)n(n+1)n - \frac{1}{5}\left(\frac{n}{3} + \frac{1}{3}\right)n\left(n + \frac{1}{2}\right)\\ &= \left(\frac{n}{5} + \frac{1}{5}\right)\left(n + \frac{1}{2}\right)n(n+1)n - \left(\frac{n}{5} + \frac{1}{5}\right)\left(n + \frac{1}{2}\right)n \cdot \frac{1}{3}.\end{aligned}$$

Ibn al-Haytham stated his result verbally in a form we translate into modern notation as

$$\sum_{i=1}^{n} i^4 = \left(\frac{n}{5} + \frac{1}{5}\right)n\left(n + \frac{1}{2}\right)\left[(n+1)n - \frac{1}{3}\right].$$

The result can also be written as a polynomial:

$$\sum_{i=1}^{n} i^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}.$$

It is clear that this formula can be used as Fermat and Roberval used Roberval's inequality to show that

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} i^4}{n^5} = \frac{1}{5}.$$

Ibn al-Haytham used his result on sums of integral powers to perform what we would call an integration. In particular, he applied his result to determine the volume of the solid formed by rotating the parabola $x = ky^2$ around the line $x = kb^2$, perpendicular to the axis of the parabola, and showed that this volume is $8/15$ of the volume of the cylinder of radius kb^2 and height b. (See FIGURE 2.) His formal argument was a typical Greek-style exhaustion argument using a double *reductio ad absurdum*, but in essence his method involved slicing the cylinder and paraboloid into n disks, each of thickness $h = b/n$, and then adding up the disks. The ith disk in the paraboloid has radius $kb^2 - k(ih)^2$ and therefore has volume $\pi h(kh^2n^2 - ki^2h^2)^2 = \pi k^2h^5(n^2 - i^2)^2$. The total volume of the paraboloid is therefore approximated by

$$\pi k^2 h^5 \sum_{i=1}^{n-1} (n^2 - i^2)^2 = \pi k^2 h^5 \sum_{i=1}^{n-1} (n^4 - 2n^2i^2 + i^4).$$

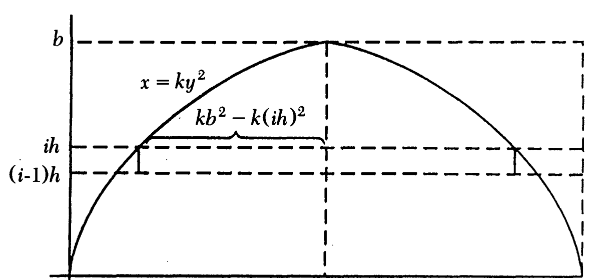

FIGURE 2

But since ibn al-Haytham knew the formulas for the sums of integral squares and fourth powers, he could calculate that

$$\sum_{i=1}^{n-1} (n^4 - 2n^2i^2 + i^4) = \frac{8}{15}(n-1)n^4 + \frac{1}{30}n = \frac{8}{15}n \cdot n^4 - \frac{1}{2}n^4 - \frac{1}{30}n$$

and therefore that

$$\frac{8}{15}(n-1)n^4 < \sum_{i=1}^{n-1} (n^2 - i^2)^2 < \frac{8}{15}n \cdot n^4.$$

But the volume of a typical slice of the circumscribing cylinder is $\pi h(kb)^2)^2 = \pi k^2h^5n^4$, and therefore the total volume of the cylinder is $\pi k^2h^5 n \cdot n^4$, while the volume of the cylinder less its "top slice" is $\pi k^2h^5(n-1)n^4$. The inequality then shows that the volume of the paraboloid is bounded between $8/15$ of the cylinder less its top slice and $8/15$ of the entire cylinder. Because the top slice can be made as small as desired by taking n sufficiently large, it follows that the volume of the paraboloid is exactly $8/15$ of the volume of the cylinder as asserted.

Ibn al-Haytham's formula for the sum of fourth powers shows up in other places in the Islamic world over the next few centuries. It appears in the work of Abu-l-Hasan ibn Haydur (d. 1413), who lived in what is now Morocco, and in the work of Abu Abdallah ibn Ghazi (1437–1514), who also lived in Morocco. (See [3] for details.) Furthermore, one also finds the formula in *The Calculator's Key* of Ghiyath al-Din Jamshid al-Kashi (d. 1429), a mathematician and astronomer whose most productive years were spent in Samarkand, now in Uzbekistan, in the court of Ulugh Beg. We do not know, however, how these mathematicians learned of the formula or for what purpose they used it.

Trigonometric Series in Sixteenth-Century India

The sum formulas for integral powers surface in sixteenth-century India and they are used to develop the power series for the sine, cosine, and arctangent. These power series appear in Sanskrit verse in the *Tantrasangraha-vyakhya* (of about 1530), a commentary on a work by Kerala Gargya Nilakantha (1445–1545) of some 30 years earlier. Unlike the situation for many results of Indian mathematics, however, a detailed derivation of these power series exists, in the *Yuktibhasa*, a work in Malayalam, the language of Kerala, the southwestern region of India. This latter work was written by Jyesthadeva (1500–1610), who credits these series to Madhava, an Indian mathematician of the fourteenth century.

Even though we do not know for sure whether Madhava was the first discoverer of the series, it is clear that the series were known in India long before the time of Newton. But why were the Indians interested in these matters? India had a long tradition of astronomical research, dating back to at least the middle of the first millennium B.C. The Indians had also absorbed Greek astronomical work and its associated mathematics during and after the conquest of northern India by Alexander the Great in 327 B.C. Hence the Indians became familiar with Greek trigonometry, based on the chord function, and then gradually improved it by introducing our sine, cosine, and tangent. Islamic mathematicians learned trigonometry from India, introduced their own improvements, and, after the conquest of northern India by a Moslem army in the twelfth century, brought the improved version back to India. (See [4] for more details.)

The interaction of astronomy with trigonometry brings an increasing demand for accuracy. Thus Indian astronomers wanted an accurate value for π (which comes from knowing the arctangent power series) and also accurate values for the sine and cosine (which comes from their power series) so they could use these values in determining planetary positions. Because a recent article [8] in this MAGAZINE discussed the arctangent power series, we will here consider only the sine and cosine series.

The statement of the Indian rule for determining these series is as follows: "Obtain the results of repeatedly multiplying the arc $[s]$ by itself and then dividing by $2,3,4,\dots$ multiplied by the radius $[\rho]$. Write down, below the radius (in a column) the even results [i.e. results corresponding to $n=2,4,6$ in $s^n/n!\rho^{n-1}$], and below the radius (in another column) the odd results [corresponding to $n=3,5,7,\dots$ in $s^n/n!\rho^{n-1}$]. After writing down a number of terms in each column, subtract the last term of either column from the one next above it, the remainder from the term next above, and so on, until the last subtraction is made from the radius in the first column and from the arc in the second. The two final remainders are respectively the cosine and the sine, to a certain degree of approximation." [6, p. 3] These words can easily be translated into the formulas:

$$x=\cos s=\rho-\frac{s^2}{2!\rho}+\frac{s^4}{4!\rho^3}-\cdots+(-1)^n\frac{s^{2n}}{(2n)!\rho^{2n-1}}+\cdots$$

$$y=\sin s=s-\frac{s^3}{3!\rho^2}+\frac{s^5}{5!\rho^4}-\cdots+(-1)^n\frac{s^{2n+1}}{(2n+1)!\rho^{2n}}+\cdots$$

(These formulas reduce to the standard power series when ρ is taken to be 1.)

The Indian derivations of these results begin with the obvious approximations to the cosine and sine for small arcs and then use a "pull yourself up by our own bootstraps" approach to improve the approximation step by step. The derivations also make use of the notion of differences, a notion used in other aspects of Indian mathematics as well. In our discussion of the Indian method, we will use modern notation to enable the reader to follow these sixteenth-century Indian ideas.

We first consider the circle of FIGURE 3 with a small arc $\alpha = \widehat{AC} \approx AC$. From the similarity of triangles AGC and OEB, we get

$$\frac{x_1 - x_2}{\alpha} = \frac{y}{\rho} \quad \text{and} \quad \frac{y_2 - y_1}{\alpha} = \frac{x}{\rho}$$

$$\text{or} \quad \frac{\alpha}{\rho} = \frac{x_1 - x_2}{y} = \frac{y_2 - y_1}{x}.$$

In modern terms, if $\angle BOF = \theta$ and $\angle BOC = \angle AOB = d\theta$, these equations amount to

$$\sin(\theta + d\theta) - \sin(\theta - d\theta) = \frac{y_2 - y_1}{\rho} = \frac{\alpha x}{\rho^2} = \frac{2\rho\, d\theta}{\rho} \cos\theta = 2\cos\theta\, d\theta$$

and

$$\cos(\theta + d\theta) - \cos(\theta - d\theta) = \frac{x_2 - x_1}{\rho} = -\frac{\alpha y}{\rho^2} = -\frac{2\rho\, d\theta}{\rho} \sin\theta = -2\sin\theta\, d\theta.$$

Now, suppose we have a small arc s divided into n equal subarcs, with $\alpha = s/n$. For simplicity we take $\rho = 1$, although the Indian mathematicians did not. By applying the previous results repeatedly, we get the following sets of differences for the y's (FIGURE 4) (where $y_n = y = \sin s$):

$$\begin{aligned}
\Delta_n y &= y_n - y_{n-1} = \alpha x_n \\
\Delta_{n-1} y &= y_{n-1} - y_{n-2} = \alpha x_{n-1} \\
&\cdots \\
\Delta_2 y &= y_2 - y_1 = \alpha x_2 \\
\Delta_1 y &= y_1 - y_0 = y_1 = \alpha x_1.
\end{aligned}$$

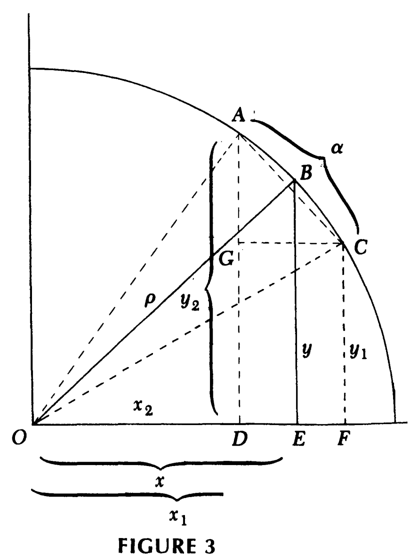

FIGURE 3

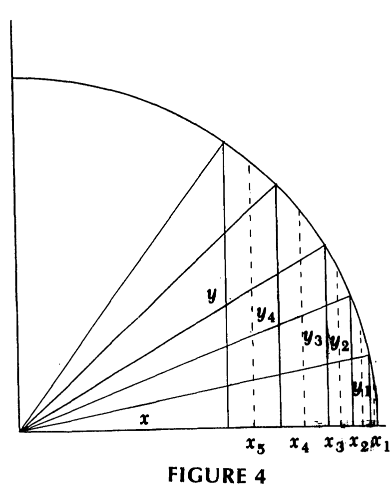

FIGURE 4

Similarly, the differences for the x's can be written

$$\Delta_{n-1}x = x_n - x_{n-1} = -\alpha y_{n-1}$$
$$\cdots$$
$$\Delta_2 x = x_3 - x_2 = -\alpha y_2$$
$$\Delta_1 x = x_2 - x_1 = -\alpha y_1.$$

We next consider the second differences on the y's:

$$\Delta_2 y - \Delta_1 y = y_2 - y_1 - y_1 + y_0 = \alpha(x_2 - x_1) = -\alpha^2 y_1.$$

In other words, the second difference of the sines is proportional to the negative of the sine. But since $\Delta_1 y = y_1$, we can write this result as

$$\Delta_2 y = y_1 - \alpha^2 y_1.$$

Similarly, since

$$\Delta_3 y - \Delta_2 y = y_3 - y_2 - y_2 + y_1 = \alpha(x_3 - x_2) = -\alpha^2 y_2,$$

it follows that

$$\Delta_3 y = \Delta_2 y - \alpha^2 y_2 = y_1 - \alpha^2 y_1 - \alpha^2 y_2,$$

and, in general, that

$$\Delta_n y = y_1 - \alpha^2 y_1 - \alpha^2 y_2 - \cdots - \alpha^2 y_{n-1}.$$

But the sine equals the sum of its differences:

$$\begin{aligned} y = y_n &= \Delta_1 y + \Delta_2 y + \cdots + \Delta_n y \\ &= n y_1 - \left[y_1 + (y_1 + y_2) + (y_1 + y_2 + y_3) + \cdots + (y_1 + y_2 + \cdots + y_{n-1}) \right] \alpha^2. \end{aligned}$$

Also, $s/n \approx y_1 \approx \alpha$, or $ny_1 \approx s$. Naturally, the larger the value of n, the better each of these approximations is. Therefore,

$$y \approx s - \lim_{n \to \infty} \left(\frac{s}{n} \right)^2 \left[y_1 + (y_1 + y_2) + \cdots + (y_1 + y_2 + \cdots + y_{n-1}) \right].$$

Next we add the differences of the x's. We get

$$x_n - x_1 = -\alpha(y_1 + y_2 + \cdots + y_{n-1}).$$

But $x_n \approx x = \cos s$ and $x_1 \approx 1$. It then follows that

$$x \approx 1 - \lim_{n \to \infty} \left(\frac{s}{n} \right) (y_1 + y_2 + \cdots + y_{n-1}).$$

To continue the calculation, the Indian mathematicians needed to approximate each y_i and use these approximations to get approximations for $x = \cos s$ and $y = \sin s$. Each new approximation in turn is placed back in the expressions for x and y and leads to a better approximation. Note first that if y is small, y_i can be approximated by is/n. It follows that

$$\begin{aligned} x &\approx 1 - \lim_{n \to \infty} \left(\frac{s}{n} \right) \left[\frac{s}{n} + \frac{2s}{n} + \cdots + \frac{(n-1)s}{n} \right] \\ &= 1 - \lim_{n \to \infty} \left(\frac{s}{n} \right)^2 \left[1 + 2 + \cdots + (n-1) \right] \end{aligned}$$

$$= 1 - \lim_{n \to \infty} \frac{s^2}{n^2} \left[\frac{(n-1)n}{2} \right]$$

$$= 1 - \frac{s^2}{2}.$$

Similarly,

$$y \approx s - \lim_{n \to \infty} \left(\frac{s}{n} \right)^2 \left[\frac{s}{n} + \left(\frac{s}{n} + \frac{2s}{n} \right) + \cdots + \left(\frac{s}{n} + \frac{2s}{n} + \cdots + \frac{(n-1)s}{n} \right) \right]$$

$$= s - \lim_{n \to \infty} \frac{s^3}{n^3} [1 + (1+2) + (1+2+3) + \cdots + (1 + 2 + \cdots + (n-1))]$$

$$= s - \lim_{n \to \infty} \frac{s^3}{n^3} \left[n(1 + 2 + \cdots + (n-1)) - \left(1^2 + 2^2 + \cdots + (n-1)^2\right) \right]$$

$$= s - s^3 \lim_{n \to \infty} \left[\frac{\sum_{i=1}^{n-1} i}{n^2} - \frac{\sum_{i=1}^{n-1} i^2}{n^3} \right]$$

$$= s - s^3 \left(\frac{1}{2} - \frac{1}{3} \right)$$

$$= s - \frac{s^3}{6},$$

and there is a new approximation for y and therefore for each y_i. Note that in the transition from the second to the third lines of this calculation the Indians used ibn al-Haytham's equation $(*)$ for the case $k = 1$. Although the Indian mathematicians did not refer to ibn al-Haytham or any other predecessor, they did explicitly sketch a proof of this result in the general case and used it to show that, for any k, the sum of the kth powers of the first n integers is approximately equal to $n^{k+1}/(k+1)$. This result was used in the penultimate line of the above calculation in the cases $k = 1$ and $k = 2$ and in the derivation of the power series for the arctangent as discussed in [8].

To improve the approximation for sine and cosine, we now assume that $y_i \approx (is/n) - (is)^3/(6n^3)$ in the expression for $x = \cos s$ and use the sum formula in the case $k = 3$ to get

$$x \approx 1 - \lim_{n \to \infty} \frac{s}{n} \left[\frac{s}{n} - \frac{s^3}{6n^3} + \frac{2s}{n} - \frac{(2s)^3}{6n^3} + \cdots + \frac{(n-1)s}{n} - \frac{((n-1)s)^3}{6n^3} \right]$$

$$= 1 - \frac{s^2}{2} + \lim_{n \to \infty} \frac{s^4}{6n^4} \left[1^3 + 2^3 + \cdots + (n-1)^3 \right]$$

$$= 1 - \frac{s^2}{2} + \frac{s^4}{6} \lim_{n \to \infty} \frac{\sum_{i=1}^{n-1} i^3}{n^4}$$

$$= 1 - \frac{s^2}{2} + \frac{s^4}{6} \cdot \frac{1}{4}$$

$$= 1 - \frac{s^2}{2} + \frac{s^4}{24}.$$

Similarly, ibn al-Haytham's formula for the case $j = 3$ and the sum formula for the cases $j = 3$ and $j = 4$ lead to a new approximation for $y = \sin s$:

$$y \approx s - \frac{s^3}{6} + \lim_{n\to\infty}\left(\frac{s}{n}\right)^2\left[\frac{s^3}{6n^3} + \left(\frac{s^3}{6n^3} + \frac{(2s)^3}{6n^3}\right) + \cdots + \left(\frac{s^3}{6n^3} + \frac{(2s)^3}{6n^3} + \cdots + \frac{((n-1)s)^3}{6n^3}\right)\right]$$

$$= s - \frac{s^3}{6} + \lim_{n\to\infty}\frac{s^5}{6n^5}\left[1^3 + (1^3+2^3) + \cdots + \left(1^3+2^3+\cdots+(n-1)^3\right)\right]$$

$$= s - \frac{s^3}{6} + \lim_{n\to\infty}\frac{s^5}{6n^5}\left[n\left(1^3+2^3+\cdots+(n-1)^3\right) - \left(1^4+2^4+\cdots+(n-1)^4\right)\right]$$

$$= s - \frac{s^3}{6} + \frac{s^5}{6}\lim_{n\to\infty}\left[\frac{\sum_{i=1}^{n-1} i^3}{n^4} - \frac{\sum_{i=1}^{n-1} i^4}{n^5}\right]$$

$$= s - \frac{s^3}{6} + \frac{s^5}{6}\left(\frac{1}{4} - \frac{1}{5}\right)$$

$$= s - \frac{s^3}{6} + \frac{s^5}{120}.$$

Because Jyesthadeva considers each new term in these polynomials as a correction to the previous value, he understood that the more terms taken, the more closely the polynomials approach the true values for the sine and cosine. The polynomial approximations can thus be continued as far as necessary to achieve any desired approximation. The Indian authors had therefore discovered the sine and cosine power series!

Conclusion

How close did Islamic and Indian scholars come to inventing the calculus? Islamic scholars nearly developed a general formula for finding integrals of polynomials by A.D. 1000—and evidently could find such a formula for any polynomial in which they were interested. But, it appears, they were not interested in any polynomial of degree higher than four, at least in any of the material which has so far come down to us. Indian scholars, on the other hand, were by 1600 able to use ibn al-Haytham's sum formula for arbitrary integral powers in calculating power series for the functions in which they were interested. By the same time, they also knew how to calculate the differentials of these functions. So some of the basic ideas of calculus were known in Egypt and India many centuries before Newton. It does not appear, however, that either Islamic or Indian mathematicians saw the necessity of connecting some of the disparate ideas that we include under the name calculus. There were apparently only specific cases in which these ideas were needed.

There is no danger, therefore, that we will have to rewrite the history texts to remove the statement that Newton and Leibniz invented the calculus. They were certainly the ones who were able to combine many differing ideas under the two unifying themes of the derivative and the integral, show the connection between them, and turn the calculus into the great problem-solving tool we have today. But what we do not know is whether the immediate predecessors of Newton and Leibniz, including in particular Fermat and Roberval, learned of some of the ideas of the Islamic or Indian mathematicians through sources of which we are not now aware.

The entire question of the transmission of mathematical knowledge from one culture to another is a matter of current research and debate. In particular, with more medieval Arabic manuscripts being discovered and translated into European languages, the route of some mathematical ideas can be better traced from Iraq and Iran into Egypt, then to Morocco and on into Spain. (See [3] for more details.) Medieval Spain was one of the meeting points between the older Islamic and Jewish cultures and the emerging Latin-Christian culture of Europe. Many Arabic works were translated there into Latin in the twelfth century, sometimes by Jewish scholars who also wrote works in Hebrew. But although there is no record, for example, of ibn al-Haytham's work on sums of integral powers being translated at that time, certain ideas he used do appear in both Hebrew and Latin works of the thirteenth century. And since the central ideas of his work occur in the Indian material, there seems a good chance that transmission to India did occur. Answers to the questions of transmission will require much more work in manuscript collections in Spain and the Maghreb, work that is currently being done by scholars at the Centre National de Recherche Scientifique in Paris. Perhaps in a decade or two, we will have evidence that some of the central ideas of calculus did reach Europe from Africa or Asia.

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REFERENCES

1. J. L. Berggren, *Episodes in the Mathematics of Medieval Islam*, Springer-Verlag, New York, 1986.
2. Walter E. Clark, *The Aryabhatiya of Aryabhata* University of Chicago Press, Chicago, 1930.
3. Ahmed Djebbar, *Enseignement et Recherche Mathématiques dans le Maghreb des XIII^e – XIV^e Siecles* (Publications Mathématiques D'Orsay No. 81-02) Université de Paris-Sud, Orsay, France, 1981.
4. Victor Katz, *A History of Mathematics: An Introduction*, Harper Collins Publishers, New York, 1993.
5. Michael Mahoney, *The Mathematical Career of Pierre de Fermat 1601–65*, Princeton University Press, Princeton, NJ, 1973.
6. C. T. Rajagopal and A. Venkataraman, The sine and cosine power-series in Hindu mathematics, *J. of the Royal Asiatic Society of Bengal—Science* 15 (1949), 1–13.
7. Roshdi Rashed, Ibn al-Haytham et la measure du paraboloide, *J. for the History of Arabic Science* 5 (1981), 262–291.
8. Ranjan Roy, The discovery of the series formula for π by Leibniz, Gregory and Nilakantha, this Magazine 63 (1990), 291–306.
9. Derek Whiteside, *The Mathematical Papers of Isaac Newton*, Cambridge University Press, Cambridge, 1967–1981.