Selections from Ibn al-Haytham (965-1041):
Treatise on the volume of the sphere.

Translation by Jan P. Hogendijk based on the Arabic edition by R. Rashid. References such as [295] are to the pages of the Arabic edition; see the bibliography at the end. Explanatory additions by Jan P. Hogendijk are in square brackets.

[295] In the name of God, the Merciful, the Compassionate.

Treatise by al-Ḥasan ibn al-Ḥasan ibn al-Haytham on the volume of the sphere.

Many things in geometry can be obtained by different approaches, and the proof of them is possible in a number of ways. Thus, one mathematician can continue to work on a matter that had been discussed by others before, and he can reach the goal, even if someone else had arrived at it before, if he finds a (new) method to discuss it which had not been found by any earlier (mathematician) before him. A number of mathematicians have talked about the measurement of the sphere, and they have established the proof for the quantity of its volume. Each of them followed a method different from the others.

Since their discussions of this matter have reached us and we have become acquainted with their proofs, we have thought deeply about the volume of the sphere - whether it can be obtained by a method different from the methods that had been used by those that have discussed the subject previously. When we spent a lot of attention on this, a new way occurred to us for finding the volume of the sphere which is more concise and shorter than all the ways which our predecessors have followed, with a clearer proof and a more evident explanation. So in this situation it is permissible for us to discuss the volume of the sphere, although a number of mathematicians have already talked about it before us.

[297] We begin this with an easy aritmetical lemma which will facilitate the understanding of our aim. It is as follows. If we consider [a finite series of] the successive integers, beginning with one, and increasing by one at a time, and if one-third of the greatest number is added to one-third, and the sum is multiplied by the greatest number, and one half is added to the greatest number, and this (sum) is multiplied by the first product, then
the result is the sum of the squares of these numbers. [in modern terms
$1^2 + 2^2 + \ldots + n^2 = \frac{1}{3}n + \frac{1}{2}(n + \frac{1}{3})].$

Ibn al-Haytham then draws an easy consequence, which can be stated in
modern terms as follows: $1^2 + 2^2 + \ldots + n^2 = \frac{1}{3}n^3 + c$ with $\frac{1}{2}n^2 < c < \frac{2}{3}n^2.$

We say: Every sphere is two thirds of the cylinder whose base is the
greatest circle in the sphere and whose height is equal to the diameter of the
sphere.

The example of this is the sphere $ABGD$ with centre $E$. Then I say: this
is two thirds of the cylinder whose base is the greatest circle in the sphere
and whose height is the diameter of the sphere.

(Proof): We let pass through the center $E$ of the sphere a plane which
will intersect the sphere in a great circle $ABGD$. In this circle we draw two
perpendicular diameters $AEG$ and $BED$. Through $B$ we draw the line $BZ$
parallel to line $EA$. Through point $A$ we draw the line $AZ$ parallel to line
$EB$, so $AEBZ$ is a rectangle. If line $AE$ is fixed and rectangle $AEBZ$ is
turned around line $AE$ until it has returned to its original position, rectangle
$AEBZ$ produces a cylinder whose base is the circle with radius line $BE$,
which is a radius of the sphere, and height line $EA$ which is also a radius of
the sphere. A circle whose radius is a radius of the sphere is a great
circle on the sphere. So the cylinder which is produced by the rotation of rectangle $BA$ around line $EA$ has as its basis a great circle of the sphere
and height the radius of the sphere. Let this cylinder be $BH$. If rectangle
$BA$ rotates around line $EA$, sector $ABE$ also rotates around line $EA$, and
if sector $ABE$ rotates around line $EA$, the rotation produces a hemisphere
whose base is the circle with radius line $BE$. 

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For if half of the circle \(ABGD\), containing \(ABG\) and the diameter \(AG\), rotates around the diameter \(AG\) until it returns to its original position, the rotation produces the sphere \(ABGD\), and the rotation of line \(EB\) produces a circle which bisects the sphere. So if the rectangle \(BA\) turns around the line \(EA\), the rotation produces a cylinder with basis a great circle on the sphere \(ABGD\), and height line \(EA\), which is a radius of the sphere \(ABGD\). (Thus) the rotation of sector \(ABE\) produces half of the sphere \(ABGD\).

Then I say: the hemisphere produced by the rotation of sector \(ABE\) is (equal to) two thirds of the cylinder \(BH\) which is produced by the rotation of rectangle \(BA\).

Proof of this: it cannot be otherwise. For if it were possible, let the hemisphere be unequal to two thirds of the cylinder \(BH\). Since the hemisphere is unequal to two thirds of the cylinder \(BH\), it is either greater than two thirds of the cylinder or less.

So let the hemisphere first be greater than two thirds of the cylinder, and let the hemisphere exceed two thirds of the cylinder by the magnitude \(t\).

We bisect \(AE\) at point \(T\), and we draw through point \(T\) line \(TK\) parallel to line \(EB\). Then \(TK\) is perpendicular to line \(AE\). We extend \(TK\) to \(L\), then \(TL\) is equal to line \(EB\). We draw through point \(K\) a line \(SKI\) parallel to the two lines \(EA, BZ\). Then \(SK\) is equal to \(KI\), since \(AT\) is equal to \(TE\). So rectangle \(KE\) is equal to rectangle \(KA\), and rectangle \(KB\) is equal to rectangle \(KZ\). If rectangle \(BA\) rotates around line \(EA\), the rectangles \(EK, KA\) produce two \([311]\) equal cylinders, and the two figures \(KB, KZ\) produce two equal round solids which circumscribe the equal cylinders. So the cylinder produced by the rotation of rectangle \(KE\) and the round solid produced by the rotation of rectangle \(KZ\) are together half of the cylinder \(BH\).

Again, we bisect \(AT\) at the point \(M\), and we draw through point \(M\) line \(MN\) parallel to line \(EB\). Then \(MN\) is perpendicular to line \(AE\). We extend \(MN\) to \(O\). Then \(MO\) is equal to line \(EB\). We draw through point \(N\) line \(WNX\) parallel to the lines \(KS, LZ\). Then \(WN\) is equal to \(NX\), and rectangle \(NT\) is equal to rectangle \(NA\), and rectangle \(NK\) is equal to rectangle \(NS\). If rectangle \(BA\) rotates around line \(EA\), rectangle \(KA\) rotates (also). Then the rectangles \(NT, NA\) produce two equal cylinders, and the rectangles \(NK, NS\) produce two round solids. The cylinder produced by the rotation of rectangle \(NT\) together with the round solid produced by the rotation of rectangle \(NS\) are together half of the cylinder which is produced by the rotation of rectangle \(KA\).
Again we bisect line $TE$ at point $F$. We draw through $F$ line $FC$ parallel to line $EB$. Then $FC$ is perpendicular to $AE$. We extend $FC$ to $Q$. Then $FQ$ is equal to line $EB$. We draw through point $C$ line $JCR^*$ parallel to the lines $ET, BL$. Then $JC$ is equal to $CR^*$, and rectangle $CK$ is equal to rectangle $CI$, and rectangle $CB$ is equal to rectangle $CL$. If rectangle $BA$ rotates around line $EA$, rectangle $BK$ (also rotates), and the rotation of it produces a round solid, and the rotation of rectangles $CK, CI$ produces two round solids, and the rotation of rectangles $CB, CL$ produces two round solids. Then the round solid produced by the [313] rotation of rectangle $CI$ together with the round solid produced by the rotation of rectangle $CL$, equal to half of the round solid produced by the rotation of rectangle $KB$.

So the (sum of the) cylinder produced by the rotation of rectangle $NT$ together with the round solid produced by the rotation of rectangle $NS$ together with the two round solids produced by the rotation of the two rectangles $CI, CL$ is half of the cylinder produced by the rotation of rectangle $KA$ together with half of the round solid produced by the rotation of rectangle $KB$. It has been proved that the (sum of the) cylinder produced by the rotation of rectangle $KE$ together with the round solid produced by the rotation of rectangle $KZ$ is equal to half of cylinder $BH$.

Since this is the case, half of the cylinder $BH$ has been removed, and half of the remainder has also been removed. If we bisect all lines $AM, MT, TF, FE$, and from the points of bisection we draw lines parallel to line $BE$, and if from their intersection points with arc $AB$ we draw lines parallel to line $AE$, the rectangles $BC, CK, KN, NA$ are all divided into four parts and (of these four parts,) each pair of opposite rectangles is equal to half of the rectangle out of which they were produced, and the round solids which are produced by the rotation of these two rectangles are half (the volume) of the round solids which are produced by the rotation of rectangles $BC, CK, KN, NA$. If this (process) is always continued, half of the cylinder $BH$ is cut off, and half of the remainder, and half of the (new) remainder. . . . (the text is unclear, probably the manuscripts are corrupted, and the editor Rashed does not understand the mathematics)

For each pair of different magnitudes, the following holds: if half of the greater of them is cut off, and half of the remainder, and if this is always continued, then it is necessary that (eventually) a magnitude will remain which is less than the lesser of the two (original) magnitudes. For if half of the (first) magnitude is cut off, and half of the remainder is cut off, (so we have been cutting) two times, more than half of the (first) magnitude
has been cut off. So if half of a magnitude is cut off, and then half of the remainder, and if we do this many times, we will cut off in every two steps more than half. For each pair of different magnitudes the following (theorem) holds: if of the greater < more than > half is cut off, and from the remainder < more than > half, and if this (operation) is continued indefinitely, then eventually a magnitude will remain which is less than the lesser magnitude (of the two). [This theorem was proved by Euclid in Book X of the Elements]

The cylinder BH and the magnitude t are two different magnitudes, and the greater of them is the cylinder BH. So if half of the cylinder BH is cut off, and half of the remainder, and half of the remainder, in the way which we have explained, and if this is always continued, then (eventually) there will remain a magnitude less than the magnitude t. [315] And if half of the cylinder BH is cut off, and half of the remainder, and half of the remainder, in the way which we have explained, then what remains of the cylinder consists of the round solids produced by the rectangles BC, CK, KN, NA and corresponding ones such that the surface of the sphere passes through their interiors.

Now let the parts at which the division of the cylinder as we have explained ends, and they are the parts which (together) are less than t, be the round solids which are produced by the rotation of the rectangles BC, CK, KN, NA. Then the parts of these round solids which lie inside the sphere are much less than the magnitude t. But the hemisphere exceeds two thirds of the cylinder by the magnitude t. So the remainder of the hemisphere, after these parts of the round solids in its interior have been removed, is greater than two thirds of the cylinder BH. But the remainder of the hemisphere after the parts of the round solids in the interior of it (the sphere) have been removed, is the sawed-off solid in the interior of the hemisphere, whose base is the circle with radius RE*, and whose top is the circle with radius MN. So this sawed-off solid is greater than two thirds of the cylinder BH.

Again: the lines AM, MT, TF, FE are equal, so each of the lines EF, ET, EM, EA exceeds its predecessor by line EF. So the ratios of the lines EF, ET, EM, EA are the ratios of the successive (natural) numbers, beginning with one, and always increasing by one.

So the squares of the lines EF, ET, EM, EA exceed one-third of the equal squares which are equal to the square of EA, and whose number is the number of lines EF, ET, EM, EA; and the excess is less than two-thirds of the square of EA, as has been proved in the lemma [which I have not yet translated]

The number of lines EF, ET, EM, EA is the number of division points
$F, T, M, A$, and the number of division points $E, F, T, M$ if we take $E$ instead of $A$. But the number of division points $E, F, T, M$ is equal to the number of lines $EB, FQ, TL, MO$. The lines $EB, FQ, TL, MO$ are equal to one another and each of them is equal to line $EB$, and $EB$ is equal to line $EA$. So the squares of the lines $EF, ET, EM, EA$ exceed one-third of the squares of the lines $EB, FQ, TL, MO$ by less than two thirds of the square of $EA$. The square of $EF$ together with the product $GF$ by $FA$ is the square of $EA$. [In modern terms $a^2 - b^2 = (a + b)(a - b)$ with $EA = EG = a$ and $EF = b$.] But the product $GF$ by $FA$ is the square of $FC$. [This is a property of the circle.] So the square of $EF$ together with the square of $FC$ is equal to the square of $EA$ [he could have concluded this directly from the Theorem of Pythagoras. Has the text been changed by a later scribe?], which is equal to the square of $FQ$. In the same way, the square of $ET$ together with the square of $TK$ is equal to the square of $EA$, which is equal to the square of $TL$. In the same way, the square of $EM$ together with the square of $MN$ is equal to the square of $EA$, which is equal to the square of $MO$. And the square of $EA$ is equal to the square of $EB$. So the squares of $EF, ET, EM, EA$ together with the squares of $FC, TK, MN$ are together equal to the squares of $EB, FQ, TL, MO$. But the squares of $EF, ET, EM, EA$ exceed one third of the squares of $EB, FQ, TL, MO$ by less than two thirds of the square of $EA$. So, by subtraction, the squares $FC, TK, MN$ are less than two thirds of the squares of $EB, FQ, TL, MO$, and the difference is less than two thirds of the square of $EA$.

So the circles whose radii are the lines $FC, TK, MN$ are less than two thirds times the circles whose radii are $EB, FQ, TL, MO$. But the ratio of the circles to the circles is equal to the ratio of the cylinders which stand on them to one another, if the heights of the cylinders are equal. So the cylinders whose bases are the circles with radii $FC, TK, MN$ and whose heights are the lines $EF, FT, TM$ are less than two thirds of the cylinders whose bases are the circles with radii $EB, FQ, TL, MO$ and whose heights are the equal lines $EF, FT, TM, MA$. But the cylinders whose bases are the circles with radii $FC, TK, MN$ and whose heights are the lines $EF, FT, TM$ are the sawed-off solid whose basis is the circle with radius $RE^*$ and whose top is the circle with radius $MN$, which is in the interior of the hemisphere. And the cylinders whose bases are the circles with radii the lines $EB, FQ, TL, MO$ and whose heights are the lines $EF, FT, TM, MA$ are the cylinder $BH$. So the sawed-off solid in the interior of the sphere is less than two thirds of the cylinder $BH$. 

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But it had been proved that this sawed-off solid is greater than two thirds of cylinder \( BH \), and this is impossible. [319] But this impossibility is a consequence of our assumption that the hemisphere is greater than two thirds of the cylinder \( BH \). So the hemisphere is not greater than two thirds of the cylinder \( BH \).

I say: the hemisphere is also not less than two thirds of the cylinder \( BH \).

If this were possible, then let it be less than two thirds of cylinder \( BH \), and let the difference between the hemisphere and two thirds of the cylinder be the magnitude \( t \). Then the magnitude \( t \) is less than the cylinder \( BH \).

If half of the cylinder \( BH \) is cut off, and half of the remainder, and half of the remainder, in the way which we have explained, then (eventually) there will be a remainder less than the magnitude \( t \). The part of the cylinder which remains after the division of it in the way which we have explained, consists of the round solids which are produced by the rotation of the rectangles \( BC, CK, KN, NA \) and corresponding rectangles, such that the surface of the sphere passes through their interiors. Let the division end with something (i.e., a remainder) less than the magnitude \( t \), and let this (i.e., the remainder) consist of the round solids produced by the rotation of the rectangles \( BC, CK, KN, NA \). Then the parts of those round solids outside the sphere are much less than the magnitude \( t \). But the hemisphere together with the magnitude \( t \) was (assumed to be) equal to two thirds of the cylinder \( BH \). So the hemisphere together with the parts of the round solids outside the hemisphere are much less than two thirds of the cylinder \( BH \). But the hemisphere together with the parts of the round solids outside it are the sawed-off solid whose base is the circle with radius \( EB \) and whose top is the circle with radius \( AW \), which (solid) circumscribes the hemisphere. So the sawed-off solid is less than two thirds of the cylinder \( BH \).

It has been shown before that the squares of the lines \( FC, TK, MN \) are less than two thirds of the squares of the lines \( EB, FQ, TL, MO \), and that the difference is less than two thirds the square of \( EA \). So if we add to the squares of the lines \( FC, TK, MN \) the whole square of \( EB \), which is equal to the square of \( EA \), the sum of the squares of the lines \( EB, FC, TK, MN \) is greater than two thirds of the squares of \( EB, FQ, TL, MO \). So the circles whose radii are the lines \( EB, FC, TK, MN \) are greater than two thirds of the circles whose radii are the lines \( EB, [321] FQ, TL, MO \). And the cylinders whose bases are the circles with radii lines \( EB, FC, TK, MN \) and whose heights are the lines \( EF, FT, TM, MA \), which are equal to one another, are greater
than two thirds of the cylinders whose bases are the circles with radii the lines $EB, FQ, TL, MO$ and whose heights are the lines $EF, FT, TM, MA$. But the cylinders whose bases are the circles with radii the lines $EB, FC, TK, MN$ and whose heights are the lines $EF, FT, TM, MA$ are (together) the sawed-off solid whose base is the circle with radius $EB$ and whose top is the circle with radius $AW$, that is the sawed-off solid which circumscribes the sphere. And the cylinders with bases the circles with radii the lines $EB, FQ, TL, MO$ and heights the lines $EF, FT, TM, MA$ are (together) the cylinder $BH$.

So the sawed-off solid which circumscribes the sphere is greater than two thirds of the cylinder $BH$. But it had been shown that this sawed-off solid is less than two thirds of the cylinder $BH$. This is impossible.

This impossibility is a consequence of our assumption that the hemisphere is less than two thirds of the cylinder $BH$. So the hemisphere is not less than two thirds of the cylinder $BH$. But it had also been shown that it is not greater than two thirds of the cylinder $BH$. So since the hemisphere is not greater than two thirds of the cylinder $BH$ and also not less than two thirds of it, it is two thirds of the cylinder $BH$. The whole sphere is twice the hemisphere, and the cylinder whose base is the circle with radius line $BE$ and whose height is line $AG$, which is the diameter of the sphere, that is twice line $AE$, is twice the cylinder [323] $BH$. So the sphere $ABGD$ is two thirds of the cylinder whose basis is the greatest circle in the sphere, and whose height is equal to the diameter of the sphere. That is what we wanted to prove.

End of the treatise on the volume of the sphere.

Source: R. Rashed, ed. Les Mathématiques infinitesimales, vol. 2, London: al-Furqān Islamic Heritage Foundation, 1993, pp. 307-323. Some errors in the edition have been corrected; the passages have been indicated by asterisks * and pointed brackets < >. Explanatory additions made by Jan H. appear in squarebrackets [ ]