

Homework 1 on Sections 1 and 2

Due Monday, Sep 8

This homework must be done individually. Use of AI tools is not allowed. Remember to follow Math department's guidelines for homework. Please write your solutions neatly. Typesetting in LaTeX is highly appreciated and encouraged.

1. Do problems 32 and 42 in section 1 of the textbook (pages 18 and 19).
2. Let A be a finite alphabet and let A^* denote the set of all sequences (or words) of finite lengths (including length 0) over A . Define an operation called concatenation defined on A^* as follows. For $\mathbf{a} = a_1a_2 \dots a_n$ and $\mathbf{b} = b_1b_2 \dots b_m$ in A^* , their concatenation is defined as $\mathbf{ab} = a_1a_2 \dots a_nb_1b_2 \dots b_m$. Determine if this operation is associative and commutative? Does it have an identity element?
3. Prove that if $\langle G, * \rangle$ is a group such that $x * x = e$ for every $x \in G$, then G is an abelian (commutative) group. Is the converse of this statement true as well?
4. Let $G = \mathbb{Q}[\sqrt{2}] := \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$. Show that G is a group under usual addition of real numbers. We say that G is a subgroup of \mathbb{R} and denote this by $\langle G, + \rangle \leq \langle \mathbb{R}, + \rangle$.
5. Given $(a, b) \in \mathbb{R}^2$, define $T_{a,b} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $T_{a,b}(x, y) = (x + a, y + b)$. Consider $G := \{T_{a,b} : a, b \in \mathbb{R}\}$ with the binary operation \circ of function composition. Is $\langle G, \circ \rangle$ a group? Is it a commutative group? What is the geometric interpretation of the map $T_{a,b}$? What is the geometric explanation for \circ being commutative or not?
6. Consider the matrices $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. These matrices are called Pauli matrices and they play a fundamental role in quantum computation. They represent the most commonly used single-qubit transformations. More specifically, I is the identity transformation, X is negation (interchanging the qubits $|0\rangle$ and $|1\rangle$ that form the standard/computational basis), Z changes the relative phase of a superposition in the standard basis, $Y = ZX$ is a combination of negation and phase change. Consider the set $G = \{\pm I, \pm X, \pm Y, \pm Z\}$ together with the usual matrix multiplication. Construct the operation table and show that this forms a group. This group is called the Pauli group. You can take it granted that matrix multiplication is associative. Is the Pauli group commutative?