Solutions to Problem of the Week 1

Correct solutions were submitted by Stillian Ghaidarov and Ben Johnson.

Congratulations!

Here are 2 possible solutions to this problem (there are still other solutions)

Solution 1: Let $f(x) = \frac{\ln x}{x}$ defined on $(0, \infty)$. Then $f'(x) = \frac{1-\ln x}{x^2}$. We find that $f' > 0$ on $(0, e)$ and $f' < 0$ on (e, ∞). Therefore, f is increasing on $(0, e)$ and decreasing on (e, ∞) and f has a global maximum at $x = e$. Hence, $f(e) > f(\pi)$, i.e., $\frac{\ln e}{e} > \frac{\ln \pi}{\pi} \implies \pi \ln e > e \ln \pi \implies \ln e^\pi > \ln \pi^e \implies e^\pi > \pi^e$ (since ln is 1-1). $\square$

Solution 2: Let $f(x) = x$ and $g(x) = 1 + \ln x$, then $f(1) = g(1) = 1$.
$f'(x) = 1 > \frac{1}{x} = g'(x)$ for $x > 1$. Therefore, $f(x) > g(x)$ for $x > 1$.

Since $\ln \pi > \ln e = 1$ (this is true since ln is an increasing function and $\pi > e$), we have $f(\ln \pi) > g(\ln \pi)$, i.e.,
$\ln \pi > 1 + \ln \ln \pi = \ln e + \ln \ln \pi = \ln(e \ln \pi)$
$\implies \pi > e \ln \pi$ (since ln is 1-1)
$\implies \pi \ln e > e \ln \pi$
$\implies \ln(e^\pi) > \ln(\pi^e)$
$\implies e^\pi > \pi^e$ $\square$