#1. \[ k(t) = E[t^\gamma] = \sum_{\gamma=0}^\infty t^\gamma (\theta^\gamma \rho^{\gamma(1-\rho)}) \frac{\theta^\gamma}{\gamma!} \]

Recall \( (x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k} \)

\[ k'(t) = \frac{d}{dt} (\theta t + (1-\rho))^\gamma = \gamma (\theta t + (1-\rho))^{\gamma-1} \cdot \theta = \theta \gamma (\theta t + (1-\rho))^{\gamma-1} \]

\[ k'(t) = E[Y] = \theta \gamma (\theta t + (1-\rho))^{\gamma-1} = \theta \gamma. \checkmark \]

#2. \( \theta = .5, \gamma = .01 \)

Thus, the interval from .48 to .52 contains values that are within 2 standard deviations. Chebychev's inequality says

\[ P\left( |X - E[X]| \leq 2\sigma \right) \geq 1 - \frac{1}{2^2} \]

Therefore, the lower bound for \( k=2 \) is

\[ P\left( |X - E[X]| \leq 2\sigma \right) \geq 1 - \frac{1}{2^2} = \frac{3}{4} \]

According to Chebychev's inequality, we would expect \( \frac{3}{4} (400) = 300 \) of the corns to have a diameter between .48 and .52.
#3 \( X \sim \text{Poisson}(7) \)

a.) \( P(X \geq 2) = 1 - P(X < 2) = 1 - P(X = 1) \)
\[
= 1 - \frac{7^0 e^{-7}}{0!} - \frac{7^1 e^{-7}}{1!}
\]
\[
= 1 - e^{-7} - 7e^{-7} = 0.9927
\]

b.) Let \( Y \) = total # of customers arriving in a 2 hour period.
Using properties of a Poisson process (which were discussed in class), \( Y \sim \text{Poisson}(\lambda = 2 \times 7 = 14) \)

\[
P(Y = 2) = \frac{14^2 e^{-14}}{2!} = 0.00008149
\]

c.) The probability, \( P(Y = 2) \), would stay the same!
Justification: Let \( X_1 = \# \) of customers arriving between 1:00 and 2:00
and \( X_2 = \# \) of customers arriving between 3:00 and 4:00

\[
P(Y = 2) = P(X_1 + X_2 = 2)
\]
\[
= P(X_1 = 0, X_2 = 2) + P(X_1 = 1, X_2 = 1) + P(X_1 = 2, X_2 = 0)
\]
\[
= 2 \left( \frac{e^{-7} \times 7^2 e^{-7}}{2!} \right) + \left( \frac{7^1 e^{-7}}{1!} \right) \left( \frac{7^1 e^{-7}}{1!} \right)
\]
\[
= 2 \left( 7^2 e^{-14} \right) + 7^2 e^{-14}
\]
\[
= 2 \left( 7^2 e^{-14} \right) + 7^2 e^{-14} = 0.00008149
\]
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4. \( f(y) = \begin{cases} \frac{1}{3} y^2 + y, & 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases} \)

a. \( \int_{-\infty}^{\infty} (cy^2 + y) \, dy = \int_{0}^{1} (cy^2 + y) \, dy \)
   \[ = \frac{c y^3}{3} + \frac{y^2}{2} \bigg|_{y=0}^{1} \]
   \[ = \frac{c}{3} + \frac{1}{2} \]

Setting \( \frac{c}{3} + \frac{1}{2} = 1 \) yields \( \frac{c}{3} = \frac{1}{2} \) or \( c = \frac{3}{2} \)

b. \( F(y) = \int_{0}^{y} \frac{3}{2} u^2 + u \, du = \frac{3}{2} \left( \frac{u^3}{3} \right) + \frac{u^2}{2} \bigg|_{u=0}^{y} \)
   \[ = \frac{y^3}{2} + \frac{y^2}{2} \]

Thus, \( F(y) = \begin{cases} 0, & y \leq 0 \\ \frac{y^3}{2} + \frac{y^2}{2}, & 0 \leq y \leq 1 \\ 1, & y > 1 \end{cases} \)

C. \( P(\bar{Y} \leq \frac{1}{2}) = F(\frac{1}{2}) = \frac{(\frac{1}{2})^3}{2} + \frac{(\frac{1}{2})^2}{2} = \frac{1}{16} + \frac{1}{8} = \frac{3}{16} = 0.1875 \)

D. \( P(\bar{Y} \geq \frac{1}{2} \mid \bar{Y} \geq \frac{1}{4}) = \frac{P(\bar{Y} \geq \frac{1}{2})}{P(\bar{Y} \geq \frac{1}{4})} = \frac{1 - F(\frac{1}{2})}{1 - F(\frac{1}{4})} \)

Again, convert to hours
\[ = \frac{\frac{13}{16}}{1 - \left[ \frac{1}{128} + \frac{1}{32} \right]} = \frac{\frac{13}{16}}{\frac{123}{128}} = \frac{13/16}{123/128} = \frac{13}{123} = 0.1048 \]
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#5. Let $R$ be the radius of the crater.

$$E[R] = 10 \Rightarrow \frac{1}{\lambda} = 10 \Rightarrow \lambda = \frac{1}{10}$$

Thus, $R \sim$ Exponential ($\frac{1}{10}$)

We want the expected area, where $A = \pi R^2$

$$E(A) = E[\pi R^2] = \pi E[R^2]$$

$$= \pi \left[ Var(R) + (E[R])^2 \right]$$

$$= \pi \left[ 100 + \left(\frac{1}{10}\right)^2 \right]$$

$$= 200 \pi$$

#6

a. False
b. True
c. True
e. False.