Introduction to Series

**Definition.** A *series* is an infinite sum of the form

\[ \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_2 + \ldots + a_n + a_{n+1} + \ldots \]

or

\[ \sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_2 + \ldots + a_n + a_{n+1} + \ldots \].

**Example 1.** \[ \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots \]

**Example 2.** \[ \sum_{n=0}^{\infty} \frac{1}{n^2 + 1} = 1 + \frac{1}{2} + \frac{1}{5} + \ldots \]

We will be interested in thinking about the following questions:

- What does it mean to add up infinitely many numbers?
- Which series converge (i.e. add up to a finite number)? Which series diverge (i.e. go to infinity)?
- How do series relate to functions and the other topics studied in Calculus?

**Example 3.** Use a geometric argument to show that

\[ \sum_{k=0}^{\infty} \left( \frac{1}{2} \right)^k = 2. \]

**Some definitions and terminology.** Let

\[ \sum_{k=0}^{\infty} a_k = a_0 + a_1 + a_2 + a_3 + \ldots \]

be an infinite series. The summand \( a_k \) is called the \( k \)-th term of the series. The sum of the first \( n \) terms of the series is called the \( n \)-th partial sum of the series, and is denoted by \( S_n \):

\[ S_n = a_0 + a_1 + a_2 + a_3 + \ldots + a_n = \sum_{k=0}^{n} a_k. \]
The definition of convergence of an infinite series involves the partial sums $S_n$.

**Definition.** If

$$\lim_{n \to \infty} S_n = S$$

for some finite number $S$, then the series

$$\sum_{k=0}^{\infty} a_k$$

converges to the limit $S$. Otherwise, the series diverges.

**Example 4.** Use the definition above to show that

$$\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 2.$$

**Example 5.** Telescoping series. Use the definition above to show that

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$$

converges to 1.

**Example 6.** Discuss the series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$  

This series is called the harmonic series. Can we use the definition to determine whether or not the series converges?

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The $n$-th term test for divergence. If $\lim_{n \to \infty} a_n \neq 0$, then $\sum_{n=0}^{\infty} a_n$ diverges.

**Example 7.** Does the series

$$\sum_{k=1}^{\infty} \frac{2k^2 - 3k + 1}{k^2 + 4}$$

converge or diverge?

**Example 8.** Does the series

$$\sum_{k=1}^{\infty} (-1)^k$$

converge or diverge?