A model of an arms race. We will develop a model of a two-nation arms race. A variation of this model was originally developed by Lewis Fry Richardson in the 1930s. Assume that we have two countries, A and B. Let \( A(n) \) and \( B(n) \) represent the amount (in some common monetary unit) spent on armaments by the corresponding countries in year \( n \). Assume that each country has some fixed amount of distrust of the other country, causing it to retain arms. We will now develop equations that relate the amount each country spends on arms in one year in terms of what they both spent the previous year.

First, let’s look at the increase in expenditures by country A, i.e. \( A(n+1) - A(n) \). We wish to construct an equation that takes into account the following information:

- If B spends a lot on defense in one year, then A spends more on defense in the next year.
- Since large expenditures will deplete a country’s treasury, large expenditures by A one year will cause smaller expenditures the next year.
- There should be a constant component (i.e. a constant independent of \( A(n) \) and \( B(n) \)) in our expression for \( A(n+1) - A(n) \) to account for inflation.

1. Construct an expression for \( A(n+1) - A(n) \) that models the scenario described above, and explain the meaning of any constants or parameters that you use.

2. Make similar assumptions about country B to obtain an expression for \( B(n+1) - B(n) \):
We now have a dynamical system of two equations, which we will learn how to deal with next week. For now, we will make some simplifying assumptions.

- Assume that the two countries have an equal amount of distrust of each other.
- Assume that the two countries’ economies are about the same.

1. Using the simplifying assumptions above, rewrite your expressions for \( A(n + 1) - A(n) \) and \( B(n + 1) - B(n) \).

2. Let \( T(n) \) denote the total expenditures on armaments of the two countries, i.e. \( T(n) = A(n) + B(n) \). Find an expression for \( T(n + 1) - T(n) \), and use it to construct a discrete dynamical system model for \( T(n) \).

3. Find the equilibrium value for this dynamical system, and discuss its stability.

4. Find a closed-form solution for \( T(n) \) in terms of \( n \).

5. Discuss (both mathematically and in terms of the arms race) various possible outcomes (corresponding to various initial conditions \( T(0) \) and various relationships between your constants).

Example. Before World War I, there were two alliances: France-Russia and Germany-Austria-Hungary. The estimated total expenditures of these two alliances were (in millions of pounds of sterling): 199 in 1909, 205 in 1910, and 215 in 1911. Thus, letting 1909 be year 0, we have \( T(0) = 199, T(1) = 205, \) and \( T(2) = 215 \). Use these values to determine the constants (or as many constants as possible) in your discrete dynamical system. Simulate the system numerically and/or graphically (or graph the closed form solution), and discuss the results. Interpret your results.