Subgroups Generated by Subsets

A cyclic subgroup of a group G is a subgroup of the form $H = \langle g \rangle = \{g^k \mid g \in \mathbf{Z}\}$, where g is an element of G. Recall that such a group can be described as the smallest subgroup of G containing g. That is,

$$\langle g \rangle = \bigcap_{\substack{g \in K \\ K \leq G}} K$$

In today's lab we wish to generalize these ideas. In particular, we will be interested in answering the following questions:

What is the smallest subgroup of a group G containing elements $g_1, g_2, \ldots, g_n \in G$? How can you describe an arbitrary element in this subgroup?

Or, more generally, *What is the smallest subgroup of a group G containing a subset $S \subseteq G$ and how can you describe an arbitrary element in this subgroup?*

Definition. Let S be a subset of a group G. Then the **subgroup of *G* generated by *S***, denoted by $\langle S \rangle$, is defined to be the intersection

$$\langle S \rangle = \bigcap_{\substack{S \subseteq K \\ K \leq G}} K$$

Note: If the set S in the definition above happens to be a finite set, say $S = \{g_1, g_2, \ldots, g_n\}$, then we normally write $\langle g_1, g_2, \ldots, g_n \rangle$ instead of $\langle \{g_1, g_2, \ldots, g_n\} \rangle$ when speaking about this subgroup.

Question 1. Explain why the definition above ensures that $\langle S \rangle$ is the smallest subgroup of G containing S.

Question 2. The subgroup generated by S could have been defined a second way, as the set of all possible products of elements in S. Indeed, if g_1 and g_2 are two elements in a subgroup of G then closure implies that the products $(g_1)^2$, $(g_2)^2$, $(g_1g_2)^2$, $(g_1g_2)^2(g_1)^3$ $(g_1g_2)^2(g_1)^3(g_1g_2)^7(g_2)^{12}$, etc.,… must also be in the subgroup.
Define the **closure of *S*** to be the set:

$$\overline{S} = \{s_1^{\alpha_1} s_2^{\alpha_2} \cdots s_n^{\alpha_n} \mid n \in Z, n \geq 0 \text{ and } s_i \in S, \alpha_i = \pm 1 \text{ for each } 1 \leq i \leq n\}$$

and prove that $\langle S \rangle = \overline{S}$.

Describing $\langle S\rangle$ as the closure of S is particularly helpful when you want to be able to describe an arbitrary element in $\langle S\rangle$. The second definition is also more easily incorporated into computer programs such as **gap**.

Question 2. Let's look at some examples in **gap**. Type in the commands below to define the subgroup S_5 generated by the two cycle (1, 2) and the three cycle (1, 2, 3).

```
gap>  G:=SymmetricGroup(5);
gap>  a:=(1, 2);  b:=(1, 2, 3);
gap>  H1:=Subgroup(G,[a, b]);
gap>  Elements(H1);
gap>  Size(H1);
```

Using **gap**'s output, classify the group $\langle$ (1,2), (1,2,3) $\rangle$.

Question 3. Use **gap** to classify each of the subgroups of S_5 listed below.

a.) H2 = $\langle$ (1,2), (2,3,4) $\rangle$

b.) H3 = $\langle$ (1,2), (3,4,5) $\rangle$

c.) H4 = $\langle$ (1,2), (1,2,3,4) $\rangle$

d.) H5 = $\langle$ (1,2), (2,3,4,5) $\rangle$

Experiment with other pairs of cycles until you are able to answer the questions that follow.

e.) Given a 2-cycle (a, b) and a 3-cycle (c, d, e) in S_5, when is $S_5 = \langle (a, b), (c, d, e) \rangle$?

f.) For which cycles, (a, b) and (c, d, e, f) in S_5, is $S_5 = \langle (a, b), (c, d, e, f) \rangle$?

g.) For which cycles, (a, b) and (c, d, e, f, g) in S_5, is $S_5 = \langle (a, b), (c, d, e, f, g) \rangle$?

Question 4. Classify the subgroups of S_5 listed below.

a.) H6 = $\langle$ (1, 2, 3), (2, 3, 4) $\rangle$

b.) H7 = $\langle$ (1, 2, 3), (3, 4, 5) $\rangle$

c.) H8 = $\langle$ (1, 2, 3), (2, 3, 4), (3, 4, 5) $\rangle$

e.) Can S_5 be generated by 3-cycles? Why or why not?

Question 5. Note that for any group *G*, we can certainly say that *G* is generated by all elements in *G*. That is, $\langle G \rangle = G$. However, in practice we are interested in finding a small set of generators for a group. If *G* is cyclic, for example, then the smallest set will contain just one element – the generator. In general, it is difficult to find a smallest set of generators for a group.

Show that the symmetric group, S_n, can be generated by just two generators. Then explain why any generating set of S_n must contain at least two elements. (Hence a minimal generating set of S_n has order 2.)