Exam 3

Math 112: Calculus B Name: ____________________

100 points 11/18/2002

- You must show all work to receive full credit.

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1) Evaluate $\int \sec^4(x)\tan^4(x)\,dx$.

Solution: $u = \tan(x) \quad du = \sec^2(x)\,dx$

$$\begin{aligned}
\int \sec^4(x)\tan^4(x)\,dx &= \int \sec^2(x)\tan^4(x)\sec^2(x)\,dx \\
&= \int (1+\tan^2(x))\tan^4(x)\sec^2(x)\,dx \\
&= \int (1+u^2)u^4\,du \\
&= \int (u^4+u^6)\,du \\
&= \frac{u^5}{5}+\frac{u^7}{7}+C \\
&= \frac{\tan^5(x)}{5}+\frac{\tan^7(x)}{7}+C
\end{aligned}$$

2) Evaluate $\int \frac{x^2+3x-4}{x^3-4x^2+4x}\,dx$.

Solution: $\int \frac{x^2+3x-4}{x^3-4x^2+4x}\,dx = \int \frac{x^2+3x-4}{x(x-2)^2}\,dx$

$\frac{x^2+3x-4}{x(x-2)^2} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$

$$\begin{aligned} A &= -1 \\ B &= 2 \\ C &= 3 \end{aligned}$$

$$\begin{aligned} \int \frac{x^2+3x-4}{x^3-4x^2+4x}\,dx &= \int \left(\frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2}\right) dx \\ &= \int \frac{A}{x}\,dx + \int \frac{B}{x-2}\,dx + \int \frac{C}{(x-2)^2}\,dx \\ &= -\ln|x| + 2\ln|x-2| - \frac{3}{x-2} + C \end{aligned}$$

3) Evaluate $\int \frac{1}{\sqrt{x^2+4x+8}}\,dx$.

Solution: $\int \frac{1}{\sqrt{x^2+4x+8}}\,dx = \int \frac{1}{\sqrt{(x+2)^2+4}}\,dx$

$u = x+2, \quad du = dx$

$\int \frac{1}{\sqrt{x^2+4x+8}}\,dx = \int \frac{1}{\sqrt{(x+2)^2+4}}\,dx = \int \frac{1}{\sqrt{u^2+4}}\,du$

$u = 2\tan(\theta), \quad du = 2\sec^2(\theta)d\theta, \quad \sqrt{u^2+4} = 2\sec(\theta)$

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2+4x+8}}\,dx &= \int \frac{2\sec^2(\theta)}{2\sec(\theta)}\,d\theta \\
&= \int \sec(\theta)\,d\theta \\
&= \ln|\sec(\theta)+\tan(\theta)| + C \\
&= \ln|\frac{\sqrt{u^2+4}}{2} + \frac{u}{2}| + C \\
&= \ln|\frac{\sqrt{(x+2)^2+4}}{2} + \frac{x+2}{2}| + C
\end{aligned}$$

4) Evaluate $\int \frac{\ln(\sin(x))}{\tan(x)}\,dx$.

Solution: $w = \sin(x),\;\; dw = \cos(x)\,dx$

$$\begin{aligned}\int \frac{\ln(\sin(x))}{\tan(x)}\,dx &= \int \frac{\ln(\sin(x))\cos(x)}{\sin(x)}\,dx \\ &= \int \frac{\ln(w)}{w}\,dw\end{aligned}$$

$u = \ln(w),\;\; du = \frac{1}{w}\,dw,\;\; dv = \frac{1}{w}\,dw,\;\; v = \ln(w)$

$$\int \frac{\ln(w)}{w}\,dw = [\ln(w)]^2 - \int \frac{\ln(w)}{w}\,dw$$

$$2\int \frac{\ln(w)}{w}\,dw = [\ln(w)]^2 + C$$

$$\int \frac{\ln(w)}{w}\,dw = \frac{1}{2}[\ln(w)]^2 + C$$

5) Evaluate $\int_0^1 \frac{1}{x\ln(x)}\,dx$.

Solution: $u = \ln(x), \quad du = \frac{1}{x}\,dx$

$$\begin{aligned}
\int \frac{e^{-x}}{1-e^{-x}}\,dx &= \int \frac{1}{u}\,du \\
&= \ln|u| + c \\
&= \ln|\ln(x)| + C
\end{aligned}$$

$$\begin{aligned}
\int_0^1 \frac{1}{x\ln(x)}\,dx &= \int_0^{\frac{1}{2}} \frac{1}{x\ln(x)}\,dx + \int_{\frac{1}{2}}^1 \frac{1}{x\ln(x)}\,dx \\
&= \lim_{a\to 0^+}\int_a^{\frac{1}{2}} \frac{1}{x\ln(x)}\,dx + \lim_{b\to 1^-}\int_{\frac{1}{2}}^b \frac{1}{x\ln(x)}\,dx \\
&= \lim_{a\to 0^+}[\ln|\ln(\frac{1}{2})| + \ln|\ln(a)|] + \lim_{b\to 1^-}[\ln|\ln(b)| + \ln|\ln(\frac{1}{2})|] \\
&= -\infty
\end{aligned}$$

Diverges

6) Evaluate $\displaystyle\int_1^\infty \frac{e^{-x}}{1-e^{-x}}\,dx$.

Solution: $u = 1 - e^{-x},\ \ du = e^{-x}\,dx$

$$\begin{aligned}\int \frac{e^{-x}}{1-e^{-x}}\,dx &= \int \frac{1}{u}\,du \\ &= \ln|u| + C \\ &= \ln|1-e^{-x}| + C\end{aligned}$$

$$\begin{aligned}\int_1^\infty \frac{e^{-x}}{1-e^{-x}}\,dx &= \lim_{t\to\infty}\int_1^\infty \frac{e^{-x}}{1-e^{-x}}\,dx \\ &= \lim_{t\to\infty}[\ln|1-e^{-t}| - \ln|1-e^{-1}|] \\ &= -\ln|1-\frac{1}{e}|\end{aligned}$$

7) Given the sequence $\{\frac{\cos(n\pi)}{n}\}_{n=1}^{\infty}$,

a) determine whether the sequence it bounded.

Solution:

$$\begin{array}{rcl} -1 & \leq \cos(n\pi) \leq & 1 \\ \frac{-1}{n} & \leq \frac{\cos(n\pi)}{n} \leq & \frac{1}{n} \\ -1 < \frac{-1}{n} & \leq \frac{\cos(n\pi)}{n} \leq & \frac{1}{n} < 1 \end{array}$$

b) determine whether the sequence is monotone.

Solution: $\{\frac{\cos(n\pi)}{n}\} = \{\frac{-1}{1}, \frac{1}{2}, \frac{-1}{3}, \ldots\}$

$\cos(n\pi)$ oscillates between 1 and -1 forcing the terms to oscillate between positive and negative values. Thus the sequence is not monotone.

c) determine whether the sequence converges or diverges. If the sequence converges, give the value it converges to.

Solution: $\lim_{x\to\infty} \frac{\cos(\pi x)}{x} = 0 \Rightarrow \{\frac{\cos(n\pi)}{n}\}_{n=1}^{\infty} \to 0$