Problem 3: Weights of Vectors

Let $\mathbb{Z}_n^m = \{(x_1, x_2, \ldots, x_m) : x_i \in \mathbb{Z}_n\}$ be the set of all vectors of length m with entries from $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$, integers mod n.

1. What is the size of the set $\mathbb{Z}_n^m$?

2. The weight of a vector $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in \mathbb{Z}_n^m$ is the number of non-zero components of $\mathbf{x}$. For example, the weight of the binary vector $(1, 0, 0, 1, 0, 0, 1)$ is 3. If we pick a random vector from $\mathbb{Z}_3^{10}$, what is the probability that its weight is 4? If you pick 100 such vectors randomly, how many do you expect to get with weight 4?

As always, show your work, fully explain and justify your answer. A solution mainly obtained by computers or calculators will not be accepted.