Jeopardy Solutions
Math 213
Spring, 2006

Basic Integrals

\$100 $\int yx^2\,dx = \frac{1}{3}yx^3 + C$

\$200 $\int \frac{x}{3\sqrt[3]{y^2}}\,dy = \int \frac{1}{3}xy^{-\frac{2}{3}}\,dy = xy^{\frac{1}{3}} + C$

\$300

$$\begin{aligned}\int \frac{3x^3 - 4x^2 + 2y^3}{x}\,dx &= \int \frac{3x^3}{x} - \frac{4x^2}{x} + \frac{2y^3}{x}\,dx \\ &= \int 3x^2 - 4x + 2y^3\frac{1}{x}\,dx \\ &= x^3 - 2x^2 + 2y^3\ln|x| + C\end{aligned}$$

\$400 $\int 5e^{-2y}\,dx = 5e^{-2y}x + C$

\$500 $\int \sin(xy) + \cos(xy^2)\,dx = -\frac{1}{y}\cos(xy) + \frac{1}{y^2}\sin(xy^2) + C$

Substitution

\$100 $\int y^3(y^4 + 3x)^{16}\,dy$. Let $u = y^4 + 3x$, then $du = 4y^3dy$.

$$\begin{aligned}\int y^3(y^4 + 3x)^{16}\,dy &= \frac{1}{4}\int u^{16}\,du \\ &= \frac{1}{4}\frac{1}{17}u^{17} + C \\ &= \frac{1}{68}(y^4 + 3x)^{17} + C\end{aligned}$$

\$200 $\int \frac{1}{\sqrt{1 + xy^2}}\,dx$. Let $u = 1 + xy^2$, then $du = y^2\,dx$.

$$\begin{aligned}\int \frac{1}{\sqrt{1 + xy^2}}\,dx &= \frac{1}{y^2}\int u^{-\frac{1}{2}}\,du \\ &= \left(\frac{1}{y^2}\right)2u^{\frac{1}{2}} + C \\ &= \frac{2}{y^2}\sqrt{1 + xy^2} + C\end{aligned}$$

\$300 $\int x\sqrt{x-5y}\,dx$. Let $u = x - 5y$. Then $du = dx$.

$$\begin{aligned}
\int x\sqrt{x-5y}\,dx &= \int (u+5y)\sqrt{u}\,du \\
&= \int u^{\frac{3}{2}} + 5yu^{\frac{1}{2}}\,du \\
&= \frac{2}{5}u^{\frac{5}{2}} + \frac{10}{3}yu^{\frac{3}{2}} + C \\
&= \frac{2}{5}(x-5y)^{\frac{5}{2}} + \frac{10}{3}y(x-5y)^{\frac{3}{2}} + C
\end{aligned}$$

\$400 **Daily Double!** $\int \sin^3(yx)\sqrt{\cos(yx)}\,dx$. First we remember that

$$\sin^2(yx) = 1 - \cos^2(yx),$$

and, peeling off two of the $\sin(yx)$'s, we get:

$$\begin{aligned}
\int \sin^3(yx)\sqrt{\cos(yx)}\,dx &= \int \sin(yx)(1-\cos^2(yx))\sqrt{\cos(yx)}\,dx \\
&= \int \sin(yx)\left(\cos^{\frac{1}{2}}(yx) - \cos^{\frac{5}{2}}(yx)\right)\,dx
\end{aligned}$$

Now we make our substitution. Let $u = \cos(yx)$. Then $du = -y\sin(yx)\,dx$.

$$\begin{aligned}
\int \sin^3(yx)\sqrt{\cos(yx)}\,dx &= -\frac{1}{y}\int u^{\frac{1}{2}} - u^{\frac{5}{2}}\,du \\
&= -\frac{1}{y}\left(\frac{2}{3}u^{\frac{3}{2}} - \frac{2}{7}u^{\frac{7}{2}}\right) + C \\
&= -\frac{2}{3y}\cos^{\frac{3}{2}}(yx) + \frac{2}{7y}\cos^{\frac{7}{2}}(yx)) + C
\end{aligned}$$

\$500 $\int \frac{e^{2x}}{1+e^x}\,dx$. Let $u = 1 + e^x$, then $du = e^x\,dx$.

$$\begin{aligned}
\int \frac{e^{2x}}{1+e^x}\,dx &= \int \frac{(e^x)(e^x)}{1+e^x}\,dx \\
&= \int \frac{u-1}{u}\,du \\
&= \int 1 - \frac{1}{u}\,du \\
&= u - \ln|u| + C \\
&= e^x - \ln(1+e^x) + C
\end{aligned}$$

Integration by Parts

\$100 $\int xye^x\ dx$. Let $u = xy$ and $dv = e^x\, dx$. Then $du = y\, dx$ and $v = e^x$.

$$\begin{aligned}\int xye^x\, dx &= xye^x - \int ye^x\, dx \\ &= xye^x - ye^x + C\end{aligned}$$

\$200 $\int y^3 + x^3 \ln(x)\, dx = y^3 x + \int x^3 \ln(x)\, dx$. For the second integral, let $u = \ln(x)$ and $dv = x^3\, dx$. Then $du = \frac{1}{x}\, dx$ and $v = \frac{x^4}{4}$. So we have:

$$\begin{aligned}\int y^3 + x^3 \ln(x)\, dx &= y^3 x + \int x^3 \ln(x)\, dx \\ &= y^3 x + \frac{x^4}{4}\ln(x) - \int \frac{x^4}{4}\frac{1}{x}\, dx \\ &= y^3 x + \frac{x^4}{4}\ln(x) - \int \frac{x^3}{4}\, dx \\ &= y^3 x + \frac{x^4}{4}\ln(x) - \frac{x^4}{16} + C\end{aligned}$$

\$300 $\int (x - 2y)\cos(x + y)\ dy$. Let $u = x - 2y$ and $dv = \cos(x + y)\, dy$. Then $du = -2\, dy$, and $v = \sin(x + y)$. So we have

$$\begin{aligned}\int (x - 2y)\cos(x + y)\ dy &= (x - 2y)\sin(x + y) + 2\int \sin(x + y)\, dy \\ &= (x - 2y)\sin(x + y) - 2\cos(x + y) + C\end{aligned}$$

\$400 $\int x^2 \sin(yx)\, dx$. Let $u = x^2$ and $dv = \sin(yx)\, dx$. Then $du = 2x\, dx$ and $v = -\frac{1}{y}\cos(yx)$. So we have:

$$\int x^2 \sin(yx)\, dx = -\frac{x^2}{y}\cos(yx) + \int \frac{2x}{y}\cos(yx)\, dx.$$

Now we do integration by parts again. Let $u = \frac{2x}{y}$ and $dv = \cos(yx)\, dx$. Then $du = \frac{2}{y}\, dx$ and $v = \frac{1}{y}\sin(yx)$. Finally, we have:

$$\begin{aligned}\int x^2 \sin(yx)\, dx &= -\frac{x^2}{y}\cos(yx) + \int \frac{2x}{y}\cos(yx)\, dx \\ &= -\frac{x^2}{y}\cos(yx) + \frac{2x}{y^2}\sin(yx) - \int \frac{2}{y^2}\sin(yx)\, dx \\ &= -\frac{x^2}{y}\cos(yx) + \frac{2x}{y^2}\sin(yx) + \frac{2}{y^3}\cos(yx) + C.\end{aligned}$$

\$500 $\int y^2\cos(y+x)\sin(-3x)\,dx$. (This is a circular integration by parts problem.) Note first that y^2 is constant, so it comes out of the integral: $y^2\int \cos(y+x)\sin(-3x)\,dx$. We will let $u=\cos(y+x)$ and $dv=\sin(-3x)\,dx$. Then $du=-\sin(y+x)\,dx$ and $v=\frac{1}{3}\cos(-3x)$. we have

$$\begin{aligned}\int y^2\cos(y+x)\sin(-3x)\,dx &= y^2\int \cos(y+x)\sin(-3x)\,dx\\ &= y^2\left(\cos(y+x)\frac{1}{3}\cos(-3x)+\int\frac{1}{3}\cos(-3x)\sin(y+x)\,dx\right)\end{aligned}$$

Now we do integration by parts again. $u=\sin(y+x)$ and $dv=\frac{1}{3}\cos(-3x)\,dx$. Then $du=\cos(y+x)\,dx$ and $v=-\frac{1}{9}\sin(-3x)$.

$$\begin{aligned}&\int y^2\cos(y+x)\sin(-3x)\,dx\\ =\;& y^2\left(\cos(y+x)\frac{1}{3}\cos(-3x)+\int\frac{1}{3}\cos(-3x)\sin(y+x)\,dx\right)\\ =\;& y^2\left(\cos(y+x)\frac{1}{3}\cos(-3x)-\frac{1}{9}\sin(-3x)\sin(y+x)+\int\frac{1}{9}\sin(-3x)\cos(y+x)\,dx\right)\\ =\;& y^2\cos(y+x)\frac{1}{3}\cos(-3x)-\frac{y^2}{9}\sin(-3x)\sin(y+x)+\frac{1}{9}\int y^2\cos(y+x)\sin(-3x)\,dx.\end{aligned}$$

Finally we solve

$$I=\int y^2\cos(y+x)\sin(-3x)\,dx = \frac{1}{3}y^2\cos(y+x)\cos(-3x)-\frac{1}{9}y^2\sin(-3x)\sin(y+x)+\frac{1}{9}I$$

for I and get

$$I=\int y^2\cos(y+x)\sin(-3x)\,dx = \frac{3}{8}y^2\cos(y+x)\cos(-3x)-\frac{1}{8}y^2\sin(-3x)\sin(y+x)+C$$

Definite Integrals

\$100 $\int_1^2 x+y^2\,dy = xy+\left.\frac{y^3}{3}\right|_{y=1}^{y=2} = \left(2x+\frac{8}{3}\right)-\left(x+\frac{1}{3}\right)=x+\frac{7}{3}$

\$200 $\int_0^1 (x+y)\sin(y)\,dx$

$$\begin{aligned}\int_0^1 (x+y)\sin(y)\,dx &= \left.\frac{x^2}{2}\sin(y)+xy\sin(y)\right|_{x=0}^{x=1}\\ &= \left(\frac{1}{2}\sin(y)+y\sin(y)\right)-(0)\\ &= \frac{1}{2}\sin(y)+y\sin(y)\end{aligned}$$

\$300 $\int_{e}^{e^2} \frac{1}{x\ln x}\,dx$. We make the substitution $u = \ln(x)$ and $du = \frac{1}{x}\,dx$. Then

$$\int_{e}^{e^2} \frac{1}{x\ln x}\,dx = \int_1^2 \frac{1}{u}\,du = \ln|u|\Big|_{u=1}^{u=2} = \ln(2) - \ln(1) = \ln(2).$$

\$400 $\int_{\frac{\pi}{3}}^{\pi} y\cos(x)\,dx = y\sin(x)\Big|_{x=\frac{\pi}{3}}^{x=\pi} = (0)y - \frac{\sqrt{3}}{2}y = -\frac{\sqrt{3}}{2}y$

\$500 **Daily Double!** $\int_0^1 x^2(1+yx^3)^3\,dx$.
We first make the substitution $u = 1 + yx^3$ and $du = 3yx^2\,dx$. Then

$$\int_0^1 x^2(1+yx^3)^3\,dx = \frac{1}{3y}\int_1^{1+y} u^3\,du = \frac{1}{3y}\frac{u^4}{4}\Big|_{u=1}^{u=1+y} = \frac{1}{3y}\left(\frac{(1+y)^4}{4} - \frac{1}{4}\right).$$

Final Jeopardy $\int \sin^5(yx)\cos(yx)\ dy$. We first make the substitution $u = \sin(yx)$; then $du = x\cos(yx)\,dy$. So we have

$$\begin{aligned}\int \sin^5(yx)\cos(yx)\ dy &= \frac{1}{x}\int u^5\,du \\ &= \frac{1}{x}\frac{u^6}{6} + C \\ &= \frac{1}{x}\frac{\sin^6(yx)}{6} \\ &= = \frac{1}{6x}\sin^6(xy).\end{aligned}$$