Instructor's Resource Manual

for use with

Closer and Closer—
Introducing Real Analysis

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About This Manual

Dear Colleague:

I have been using drafts of this text in my own classes for some years and I have received detailed feedback from other instructors, from a variety of different sorts of institutions, that have used portions of the book in their classes. With this information in hand, I was able to make improvements in the book that make things run more smoothly for the instructors that teach from the book and for the students who learn from it. Moreover, I have gained some insight into the day-to-day issues that arise in the classroom. Because I think the usefulness of an instructor's resource manual is directly proportional to how well it addresses those day-to-day concerns, my intention is to write a practical guide that will help you when you use *Closer and Closer* in the classroom. Therefore, I write this guide almost entirely from the point of view of a teacher who has used *Closer and Closer* and only occasionally from the point of view of the author.

The Instructor's Resource Manual is divided into three major parts. I will start with the "big picture": a look at the goals for my real analysis course and the philosophical principles that I use to achieve them. *Closer and Closer* is written in such a way that its ultimate worth as learning tool depends a great deal on the way that students interact with it, with each other, and with you. Therefore, in the second part of the IRM I will speak in detail about the way that I organize and run the class when I use the book. The discussion in the second part will concentrate on the particular model that I have used for running the class: an inquiry-based learning model in which there is very little lecturing. Class time is spent in discussion, small group work, and with students presenting their proofs and solutions to one another. The instructor acts primarily as a moderator. It makes sense to think of this as the "native" model for using the book. However, it is not at all difficult to see ways of including more lecturing, if that makes more sense for your course. Even if your class is organized very differently from mine, I am confident that some of my comments will be useful to you. In the third part, I will make some general comments about the book and go through, chapter by chapter, giving a (very) brief overview of the content and the problems, as I see them.

At the end of the IRM, I include a list of the errors that I have found in the book. As I become aware of other errors, I will update this section of the IRM and users will always be able to access an up-to-date version of the errata from my personal website. Known errors will be corrected in subsequent printings of the book. If you find any errors that are not on the list, I would consider it a great kindness were you to let me know about them.

Please understand that what you find herein are thoughts based on my own experiences teaching out of *Closer and Closer*. I am continually learning new things as I teach, and therefore I have no pretensions that what I write is the definitive or final word. I hope that many of my comments will be useful to you. There is no doubt that some of them will not be. If, in using the book, you encounter difficulties that I do not address in this guide, I hope that you will feel free to contact me. I will try to help if I can. Better yet, if you find additional strategies that work in your classes or for your students, I would be grateful if you would let me know of them. That way I can use them myself and pass them on to other colleagues who are using the book.

Respectfully,

Carol S. Schumacher
July, 2008

Part I
The Big Picture

Goals

Analysis is the branch of mathematics that allows us to describe limiting processes precisely. Thus my real analysis course focuses on the rigorous, mathematical treatment of limiting processes. The overarching learning goal is for my students to understand the mathematics that governs convergence (broadly conceived) and be able to reason about the most important mathematical structures that arise from limiting processes. Students who complete the course should be well-prepared to undertake more advanced studies in mathematical analysis and to use the tools of real analysis to support their work in other branches of mathematics. Some specific curricular goals, of course, underly this more general set of goals.

- Because the emphasis is on *real* analysis, students need a rigorous understanding of the real number system, especially the least upper bound property and its uses.
- Students should be able to formulate and reason from definitions for several mathematical structures connected to limiting processes, e.g. limits of sequences, continuous functions, differentiation, integration, numerical series, ...
- In the midst of abstract formulations and theorems, students must understand how the theory makes precise the intuitive ideas they learned in their calculus courses.
- Students should see how the general theory plays itself out in concrete spaces such as $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$. Moreover, they should understand how arithmetic and order "weave" themselves into the theory of limiting processes on $\mathbb{R}$. Ideally they should also see several instances in which the linear structure of $\mathbb{R}^n$ affects the theory in higher dimensional spaces.
- Students should gain a basic understanding of the connections between geometry and analysis. For instance, they should have a sense for how open and closed sets, compact sets, complete sets, and connected sets play a role in the results of real analysis.
- In order to understand the importance of the basic results in real analysis, students should see at least one substantial "application" of real analysis. (I use the word application in a loose sense, to mean that students should see the way that the theory underlies and supports our study of topics such as differentiation, integration, sequences and series of functions, etc.)

- In a two semester sequence, students should be exposed to more and deeper "applications." In addition to all those mentioned in the previous goal, some examples might be the calculus of functions of several variables, power series, differential equations, the implicit function theorem.

Student Autonomy

I believe that our fundamental aim as teachers should be, ultimately, to make our students independent of us. Nothing they can learn in our classes will be half so valuable as the ability to attack problems and draw conclusions, to think critically, to make logical connections, and to express their ideas in clear, persuasive language, both in writing and orally. These are the skills that make a mathematician. But they really serve *all* of our students, most of whom will not become professional mathematicians. Though the vast majority of our students will forget the Heine-Borel theorem (or whatever!) almost before the ink is dry on their final exams, if we teach them to think for themselves, it lasts them for a lifetime. And it will be valuable to them no matter what they end up doing.

One thing that is necessary if we are to make our students autonomous is to give them the chance to explore mathematical ideas on their own and, more importantly, at their own pace. It is absolutely clear that we can cover more material by lecturing in class day in and day out. But it is not at *all* clear what is meant by the word "cover" in this sentence. Of course it implies that certain ideas have been paraded before our students' eyes, but what has actually been learned is debatable. It is vital to ask ourselves how often, with even the most carefully planned, beautifully crafted lectures, we are simply passing on yet one more superficial look at mathematics in which crucial mathematical details (like hypotheses and logical connections) can be ignored in favor of the "general gist" that comes at the end.

I believe we all want our students to experience the unmistakable feeling that comes when one really understands something thoroughly. Much "classroom knowledge" is fairly superficial, and students often find it hard to judge their own level of understanding. For many students, the only way they know whether they are "getting it" comes from the grade they make on an exam. By passing beyond superficial acquaintance with some mathematical ideas, students will become less reliant on such externals. When they can distinguish between really *knowing* something and merely knowing *about* something, they will be on their way to becoming independent learners.

So we must balance our students' need to see fundamental mathematical ideas with a realistic assessment of what it takes to actually learn them. And, beyond that, what it takes to turn our students into autonomous thinkers and learners. This will, of course, always be a balancing act, especially in a course like Real Analysis where there are some deep, fundamental ideas that students must understand in order for us to feel as though we have really introduced them to real analysis, as a discipline. It may be that there are some topics in which we are content to let our students see the big ideas and "get the gist." But, in general, I believe it makes sense to curb our ambitions for covering material in favor of letting our students gain a deeper understanding of fewer ideas. In the midst of these competing demands, I do my best to find a way to craft my classes so that I can help my students find their own answers rather than answering their questions for them. I wrote *Closer and Closer* with this in mind.

Philosophy

The curricular goals outlined above are very ambitious. In my experience, they are too ambitious. As always, we would like to accomplish far more than time allows, so we must choose our battles carefully. I find that I can accomplish most of my curricular goals in a two-semester sequence, but that I can accomplish many fewer than I would like with only one semester. So I have to make choices. From my point of view, the focus should be on fostering my students' precise understanding of limiting processes. As instructors we cannot completely ignore the need to cover this or that topic, this or that theorem. But I am convinced that if, in the end, my students can reason like analysts, they will be well served by my analysis course. Indeed, I find that it is so. Would I rather be able to cover a few more topics during my first semester course? Yes. But the syllabus that I cover is, nevertheless, fairly ambitious. Moreover, I have great confidence that when students who have finished my course move into a situation where they need to use real analysis (not as a set of topics, but as a way of thinking), they will be able to do so. If they missed a big theorem or two, they can pick it up, read about it, think about it, and do so in a fairly sophisticated way. I see amazing results in students in my Real Analysis II course. They have the tools they need to delve into deeper topics in analysis and to understand them. They can craft complicated, multi-stage proofs that require precise analytical thinking and present them clearly. With this preparation, they make amazing *curricular* strides.

In the preface to the book, I briefly discussed my reasons for setting the discussion of real analysis in the context of general metric spaces. In particular, I think that abstraction is good for clarifying the structure of mathematical connections. And, most will, no doubt agree, at some level. But I know that some are skeptical that the abstract approach is best for students who are first learning to think about limiting processes—"it is just too hard," they say, "for most students." But, paradoxically, I believe that there are a number of ways in which it is *easier*.

How many of us have heard, "I completely understand it when you prove something, but when I try to do it myself, I am completely lost"? In teaching students to prove theorems over the years, I have come to realize that we mathematicians are very adept at taking the many tools at our disposal and choosing the right one at the right time for proving a particular theorem. In contexts that are straightforward for us (everything we are teaching, presumably), we do this almost without realizing it. But those "straightforward" or even "trivial" situations are difficult for our students because they see a cloud of possibilities and have no idea where to focus their attention. (The picture I have in my head is that of a confident juggler who has many balls in the air at once. She knows exactly when to catch which ball and which direction to throw it in. A novice juggler rapidly loses track and everything falls apart.) This is why just showing a student how to prove a theorem and hoping he will be able to prove the next one doesn't really work. It is much more effective to find ways of focusing the student's attention on the right things and letting him construct the arguments.

This is where the metric space approach helps. The preface of the book states that "the real line is, paradoxically, too rich in mathematical structure." When I wrote that, I was making a theoretical point about understanding which mathematical ingredients are necessary for thinking about certain mathematical structures and ideas. But there is a pedagogical point to be made, as well. In a metric space there is only one tool. You can measure distances. That's it. Because this is all that is needed for the most basic analytical arguments, stating things in the context of a general metric space has the benefit of focusing the students' attention on the right thing! We set our students to using this one tool first and gradually add more tools (arithmetic, order, functions) as they become more adept at thinking as analysts. There is another thing that I believe helps. For most elementary analytical arguments, pictures in $\mathbb{R}^2$ are often easier to draw and more enlightening than pictures in $\mathbb{R}$—the extra dimension gives you more space to see what is truly going on. This also helps the students focus their attention in productive ways.

As for the question of difficulty, a huge proportion of the theorems in an elementary elementary analysis course are proved in *exactly the same way* whether they are stated in a general metric space or whether they are stated only for the real line. So the worry about difficulty is not really about the mathematics. The legitimate concern is that our students will become lost in the fog of abstraction and not have any idea what they are reasoning about. This lack of intuition is, in turn, a barrier to making good arguments. Thus, once again, we find that we must help our students focus their attention productively. Our pedagogical emphasis must be on keeping our students grounded. "In the midst of abstract formulations and theorems," we must make sure our students "understand how the theory makes precise the intuitive ideas they learned in their calculus courses." And "students should see how the general theory plays itself out in concrete spaces such as $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$." It helps to routinely discuss the significance of theorems that are being proved. But we must also make it a frequent habit to ask our students to interpret a particular result as a statement about the real numbers. We must teach them to draw pictures that illustrate a particular theorem in $\mathbb{R}$, $\mathbb{R}^2$ or in the context of Calculus I-type functions. And so forth ...[1]

[1] It is easy to fall into the trap of thinking that helping students make connections between

In conclusion, my philosophy for the course is that the emphasis must be on teaching the students to think like analysts. I believe that the metric space approach is not only more powerful, the results more general, and the knowledge more "extendable," it is a good way of separating out unrelated issues, so that students can deal with them one at a time. This makes the arguments cleaner and the final connections clearer, and this in turn supports the main goal, which is to train students to think like analysts. Abstraction is a powerful tool in mathematics. But it is useful only in as much as it talks about something we care about, so in the ebb and flow of the class we must continually make explicit the connections between abstract theory and the important contexts to which it speaks.

A New Way of Thinking

Mathematicians and Scientists were using the Calculus to solve problems for well over a century before anyone felt any need to rigorously explain what was going on. Once the need became apparent, sometime in the early 19^{th} century, it took most of another century to get the theory of limiting processes well under control. It required a completely new way of thinking about mathematics. It is this unique way of thinking that we are working to teach our real analysis students. At the heart of it all is the correct usage and interpretation of quantifiers.

By this point in their mathematical training, most students have begun to understand that the word definition means something different to mathematicians than it does to the rest of the world. Everyone else thinks that a definition is a statement that we use to understand the meaning of the word being defined. Moreover, the definition is discarded as soon as we understand what the word means. To the mathematical community, a definition is a tool that is used to make an intuitive idea precise and subject to rigorous discourse. Though a mathematical definition *may* help us to understand the concept, that is not its purpose! Nevertheless, it is hard to blame our students for the skeptical (shocked?) looks they give us when we "define" continuity by saying that

> A function f is continuous at the point a if for all $\epsilon > 0$, there exists $\delta > 0$ such that if the distance from x to a is less than δ, then the distance from $f(x)$ to $f(a)$ is less than ϵ.

abstract theorems and concrete examples is mostly necessary for students who have trouble dealing with abstraction. But personal experience leads me to believe all students learn a great deal when we force them to make connections between abstract statements and concrete situations. As an undergraduate, I was really great at reasoning my way through an argument. I *hated* having to think about examples; it seemed like busy work, a distraction from the important game that was afoot. But, like everyone else, I hit a point in graduate school where the theorems were hard enough that I could no longer clinch a proof by playing logical games. And, at that point, I had no tools for building the intuition that would give me the insight I needed to proceed. I would certainly have benefitted from taking a course in which I was required to construct examples and interpret theorems in concrete settings.

And it is definitely not hard to see why it took decades to go from an intuitive understanding of the concept of continuity to a rigorous definition for it.

As it took Gauss and Cauchy and Weierstrass and Riemann decades to make these ideas precise, we should not take it for granted that our students will immediately be able to move beyond the quagmire of quantifiers to a deep understanding of important theorems in real analysis. This is precisely why I believe the emphasis in a first analysis course needs to be on learning to think like an analyst rather than on a list of "canonical" theorems that need to be covered. There is no way to short-circuit the learning process that must take place in order for students to

- be able to understand and interpret the meaning of a statement involving stacked quantifiers.
- be able to prove a theorem in which they must establish the truth of a statement involving stacked quantifiers
- be able to use a hypothesis that involves stacked quantifiers
- be able to negate a statement involving stacked quantifiers. Students find this to be especially tricky if the stacked quantifiers end in an "if ..., then ..." statement. (The definition of continuity, for instance—a good pedagogical reason for having the convergence of a sequence be the first limiting process that is considered!)

Furthermore, once students start negating statements that begin with "for all $\epsilon > 0$, there exists ..." we end up with statements that begin with "there exists $\epsilon > 0$ such that for all ..." And these are handled completely differently in proofs! It would be easy to write a treatise on the cognitive processes at work here. I will merely point out that there are a lot of conceptual pitfalls in all of this. If we can get our students past the conceptual roadblocks inherent in the mathematical theory of "closeness," the subject matter follows. Without conquering these conceptual issues, the student cannot gain a meaningful understanding of the important theorems of analysis.

Part II

A Course that uses *Closer and Closer*

Because *Closer and Closer* has some unusual features, it may be useful for me to describe what I do when I use it in my own classes. Thus, this chapter describes how I have used *Closer and Closer* in a junior/senior-level course at Kenyon College. I include fairly detailed descriptions of the day-to-day routine of the class, various problems and pitfalls I have encountered, and my general strategies for dealing with them. The students who take the course have all had three semesters of calculus and an "introduction to proofs" course. Many have had at least one other abstract mathematics course. We encourage students to wait until at least their junior year to take the course, but some of our very best students take the course their sophomore year and do well. The course is thought to be challenging by pretty much every student who takes it.

Course Mechanics

Student Responsibilities

Preparing for Class

Since I seldom lecture, it is the students' regular responsibility to read the textbook carefully. As I mention in the note to the student at the beginning of the book, the reading is punctuated by "exercises." These exercises are meant to be the students' contribution to the reading. My students know that they are expected to read with pencil and paper in hand, and to stop and work out the exercises as they come to them. Some exercises are more challenging than others, but most are very straightforward and meant to help students understand definitions, or tease out some simple nuances in the ideas. There are a few exercises in the text that we come back to during the class period. Most are just there as part of the reading.

I assign explicit problems for the students to work on outside of class. Though I know that not every student will solve all of these, I expect that each student will regularly be proving theorems and will have worked on every assigned problem enough to understand its statement, to have in mind the relevant definitions, and to comprehend the mathematical issues at hand. Quite frankly, this sometimes works better than others—it works better with some students than with others and it works better with some classes than with others. Thus I continually work to "tweak" the pacing of the assignments, and the way that I convey my expectations to the students.

Class routine

When beginning a new topic, we may go over exercises during class. In the process, we can conveniently discuss and clarify new definitions or ideas.[2] Though I may have a specific goal in mind, I try to *moderate* rather than *lead* the discussion. I often call on students by name in order to keep everyone engaged. When the exercises in the section are a bit more challenging, I know that not all of the students will have quite gotten the message they were meant to convey. In these cases, the discussion often works better if the students are in small groups. I break up the class into groups of 3-5 students and give them a specific task to work on. This gets the students thinking together and keeps everyone involved; We may then come back together as a class to draw some "morals." Or not.

Another typical day will be taken up by students presenting their solutions/proofs to the class. This is usually done by volunteers, who receive credit for their presentations. Less frequently I assign certain problems to certain small groups of students.

Assigned Problems

When I assign work to my students, I distinguish between "class" problems and "notebook" problems. The class problems are presented in class and the notebook problems are written up and handed in. These are disjoint sets of problems, and each sort of problem has a unique role to play in the class. Students are invited to come talk to me problems outside of class, individually or in groups, if they wish.

I have no hard and fast rules about this, but generally speaking, the class problems come in three types. Along with the reading assignment, before any discussion of the topic, I assign some fairly easy "warm-up" theorems that are especially good for helping the students understand and see how to use a new definition or a new tool. Afterwards, I may assign some slightly more difficult problems that are good for generating discussion about the main ideas that are presented in the section. Finally, I make sure that the proofs of the most important theorems are presented in class. These sorts of problems, taken together, make lecturing on the ideas unnecessary, as the nuances of both the concepts and the mathematical details get teased out in the discussion that arises during the presentations.

Students explore the ideas further in the notebook problems. Notebook problems may include some specialized results, (e.g. the convexity of balls in $\mathbb{R}^n$, the interior or boundary of a set) or exploration of a tangential idea (e.g. perfect sets, cells in $\mathbb{R}^n$). I may assign some result that is useful and interesting but not central to the theory (e.g. uniformly continuous function preserve Cauchy sequences). Occasionally, I will assign the proof of a main

[2]This is needed for some sections and not for others. Sometimes the reading is self-explanatory, and we may launch directly into talking about the assigned proofs. Sometimes there are subtle points that students won't necessarily get on first reading through the material.

result that doesn't really need much discussion in class (e.g. the limit of the quotients of two convergent sequences is the quotient of the limits). If there is a more difficult proof that requires extended thinking and/or one that I think everyone in the class needs to understand thoroughly, I will often assign it as a notebook problem.

Class Presentations

Though the atmosphere in the Real Analysis class is informal and friendly, what we do in the class is serious business. In particular, the presentations made by students are taken very seriously. Here are some of the things my students need to know about making a presentation at the board:

- The purpose of class a presentation is not to prove to me that the presenter has done the problem. It is to make the ideas of the proof clear to the other students.
- In order to make the presentation go smoothly, the presenter needs to have written out the proof in detail and gone over the major ideas and transitions, so that he or she can make clear the path of the proof to others. Students should avoid just copying their solutions from their notebooks, though they may use their notebook as a reference.
- Generally speaking, presenters are to write in complete sentences, using proper English and mathematical grammar.
- Whenever possible, the presenter should draw a detailed diagram that illustrates the result and/or the proof of the result.
- Ordinarily, presenters will explain their reasoning as they go along, not simply write everything down and then turn to explain. (Though I sometimes have one or more students write up a result while another writes and presents.)
- Fellow students are allowed to ask questions at any point and it is the responsibility of the person making the presentation to answer those questions to the best of his or her ability.
- Since the presentation is directed at the students (not the instructor!), the presenter should make frequent eye-contact with the students in order to see how well the other students are following the presentation.
- Making mistakes is par for the course and will occur frequently. Small mistakes in a proof are often "fixable" on the spot with the help of other students. Everyone learns from the flaw in the reasoning. In fact, it frequently happens that everyone learns *more* than they would have with a flawless presentation! If a student presents a solution that is simply incorrect, I always give the presenter a chance to work on the problem before the next period and present it again. The student is invited to

work on the proof and then come see me before presenting again, if he/she wishes. There is no grade penalty.

Perhaps more subtle are the responsibilities of the students who are not presenting. During a class presentation, the rest of the class is not off the hook just because another student is at the board. A student presenting a problem is not a "substitute teacher," a replacement for a seasoned lecturer. Those who are sitting down are, nevertheless, expected to be active participants in the presentation. Unfortunately, I find that my students are, initially, reluctant to say anything when someone else presents something at the board. There is a sense that this amounts to a personal attack on the presenter, which is completely taboo in our student culture. So I emphasize a communal spirit.

- Active participation implies that everyone is meant to help the person at the board explain things as clearly as possible. It is a class project. Students are all in it together.

- Questions and comments should not just come from students who are confused. (Though if students *are* confused they should feel free to say so. Unfortunately, this is a hard habit to foster; I continually try to come up with strategies for encouraging it.) All students are responsible for contributing questions and comments that help clarify what is being presented at the board.

- Active participation is not just about critical comments or suggestions for improvement. When a student presents a flawless proof, students will have no suggestions to make, so they tend to just sit there, impassively. Students may *think* that they would prefer not to have their fellow students say anything about their presentations, but actually this is actually pretty awful. (How many of *us* have faced the blank stares of students in our classes? Ugh!) I encourage my students to smile, nod, give a thumbs up, or say "great job" if they are happy with the presentation.

Cooperation and Competition

Just as I foster a community spirit in the classroom, I encourage my students to work together outside of class. The kind of material that they encounter lends itself very well to give and take, and students benefit from being able to bounce ideas off of each other. I think that most students who thrive in the course are part of a small group of 2-4 students who work together regularly outside of class. As an added benefit, I think that students who work in a small group typically enjoy the class more. The intense working sessions cement friendships that go beyond mathematics.

Students are not, of course, allowed to work together on exams. They are also not permitted to write up their notebook problems together. In the handout that I prepare for the first day of class, I say explicitly that "all written work must finally be [the student's] own expression." This prevents a weaker student from relying too much on a friend who is a stronger student. Students know that after talking things out with their friends they will have to write solutions up on their own; therefore, they must thoroughly understand them. (Some students ignore this instruction at first, but if I see papers that look too similar, I remind them of it. This usually solves the problem.)

In addition to encouraging ongoing collaborations, it is can also be healthy to foster a bit of competition. I have heard of instructors who give a weakly prize for the best presentation; though I have never tried this myself, I might. Having students vying to be the first to solve a tricky problem can also add a little spice to the class. One could offer some sort of minor prize for the solution to a problem that is "hanging." Of course, one need not actually offer prizes. Throwing down the carefully chosen gauntlet every once in a while can also have a positive effect. However, it is good to be wary of crossing the line into a less healthy environment where students who are not as competitive feel marginalized or intimidated by students who are more competitive.

My Role

This is a hard section to write because it is hard to describe what I do with any great degree of precision. Clearly, there are some concrete decisions that I make as the instructor of the course. I decide:

- what topics to cover,
- what specific problems and theorems to assign
- which problems (word used broadly!) are to be discussed orally, which are to be presented at the board, which should be part of a written assignment.
- how many tests there will be and when will be given.

On a day-to-day basis, I have to be aware of the nuances that are likely to crop up in the problems I have assigned to my students so that if students are stuck on something, I can give useful hints, help them draw a useful picture, or call attention to a specific theorem or proof that they might think about.

But beyond the mundane, it is harder to pin down my role precisely. Student participation is the moving force in my real analysis class; thus my role consists largely of *responding* to what my students do and say. I have to be on my toes, continually evaluating the situation to see when to prod my students to work more diligently and when slow down and let them process subtle new ideas. I have to determine when to speak and when to keep quiet. (And anyone who knows me knows that this is *very* difficult, indeed!) I have to see when to step in to lead a "big picture" discussion. I have to know when to let the niggling details of a particular argument occupy the attention of the class. I am guided by instinct and experience.

This all amounts to a sort of "shepherding" of the class. Pacing is especially important. There are some sections that students should be able to work through quickly if they put their minds to it, and I may have to prod them to get them moving. There are some sections that take more time, either because there are more problems to work on or simply because the ideas are more subtle and need some time to sink in. After having taught real analysis for many years, I now have a daily syllabus that works pretty well, but when I first started teaching the class it was sometimes hard to judge just what the correct pacing should be. (If you would like to see my daily syllabus and other course information, it is all available online. Either google Carol Schumacher and follow your nose, or feel free to e-mail me and I will send you a link.)

Though I like to have my students prove the theorems on their own, it is unproductive to let them wrestle forever with any one difficult issue. I have to pay close attention to the students so that I know when they can overcome an obstacle on their own and when an additional hint might be necessary. Of course, how readily I give a hint will also depend on the specific topic at hand—some things are worth more wrestling than others! There are topics that can be left "hanging" for further student thought while the class moves ahead, whereas other obstacles must be surmounted before any further progress can be made.

In the classroom—FAQ

How do I choose who goes to go to the board? Recently, my approach has been to ask for volunteers. I find out which students have solutions for which problems and, with this in mind, divide up the presentations as democratically as possible. (I have tried other things, and I vary my practice from time to time.) If two or more students have the same problem, I ask the student who has presented the fewest times during the semester to present his/her solution. If the students have presented exactly the same number of times, I will call on the person who has presented least recently. If one or more students have several problems, the student who has presented the least gets first choice, and so on.

<u>Other schemes</u>: I use a volunteer system because it is simple and the least stressful for the students, and it seems to work pretty well for me. However, some instructors prefer to call on students by name. Either they choose the student themselves or use some randomization device for deciding who will be called on. There are clearly advantages and disadvantages in all possible schemes. Calling on students increases the chances that all students will go to the board about equally often. In a volunteer system, it is likely that some students will present solutions more often than others. Moreover, calling on students builds in a "fear factor." Students who know they may be called on by name are less likely to come to class unprepared. There must, of course, be some mechanism for "passing," in case the student who is called on hasn't solved the problem that is requested. Some instructors let students pass whenever they wish, with no penalty. Other instructors keep track of whether and when a student passes. Students are allowed a certain number of "free" passes during the semester or per

week, or whatever; too many passes count against the student's final grade. But "cold calling" has disadvantages, as well. In general, having students present the problems they feel confident about makes things run more smoothly. It may be that student A is very confident about her solution to problem 1 but is less confident about her solution to problem 2, whereas student B is not very confident about his solution to problem 1 but feels his solution to problem 2 is really solid. It makes sense for student A to present problem 1 and for student B to present problem 2. This is more likely to happen if the instructor uses volunteers.

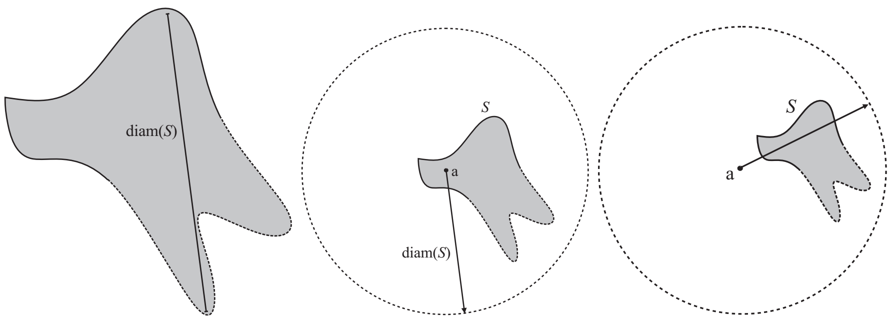
What about instructor's choice vs. randomization? Random selection keeps the students on their toes. If the instructor chooses which students to call on and "spreads the wealth," a student who presents on Monday may feel "safe" to come to class unprepared on Wednesday. Moreover, randomization keeps students from feeling as though they are being "picked on" by the instructor. For instance, it prevents students from getting the impression (true or not) that they are always asked to prove harder theorems while others get by presenting easier ones. On the other hand, if an instructor just chooses whom to call on, it is possible to "fine tune" who presents what. For instance, if there is a particularly tricky problem coming up, the instructor can take aside a student that needs to be challenged and strongly suggest that that student take it on and present it. If there is a very shy student, the instructor can let that student practice a presentation privately and then call on him/her to present the problem. If the instructor knows that a particular student has been struggling, but has finally managed to conquer a problem he/she is very proud of, the instructor can call on that student when it comes time to present it. (I can "fudge" some of the fine-tuning with my volunteer scheme, but not as easily.)

What do I do during class presentations? The hardest thing for me is to be patient, keep my counsel, and let the students do the talking. It is tempting to want to hurry things along, but it is best to fight the impulse. Letting students work through things at their pace always takes longer, but they come away with a better grasp of the ideas. That, after all, is the whole point! Moreover, students need to learn to talk to each other more than they talk to me. When conversing with me, students inevitably look to me for answers. In a conversation with their peers, students know they have to find their own answers. Fostering this sort of conversation requires patience on my part. Frequently my students will wrestle with an issue that I could rapidly clarify with a few words or a well-chosen example. However, the most valuable class meetings are those in which I manage to rein in my impulse to jump to their rescue. The students begin to talk to each other as they try to resolve the issue. Sometimes they succeed and sometimes they do not, but they learn a lot from each other and from their own struggles to state their points of view convincingly. If the discussion becomes unproductive or deadlocked or if something more needs to be said, then I can still add my own remarks to the conversation.

At times there will be some point that should be brought up but isn't forthcoming. Rather than asking questions of the person presenting, I try to call on other students, asking them questions that may bring the issue to light. In fact, I think it is always a good idea to bring up questions with students. "Jody, what do you think about the proof? Is it right?" Other good sorts of questions: "Do you see anything that is missing?" "What is it that justifies the conclusion drawn in the fourth sentence?" It is important to ask these sorts of questions when the proof is completely correct, when it is almost correct, and when it is completely wrong. (I have to confess that I am not as good about this as I should be, but the principle is certainly right!)

It probably goes without saying that learning how to make useful drawings is extremely helpful in learning to think like an analyst. A good drawing can help us gain insight into the mathematics and and, later, it can help us explain a theorem and/or its proof to others. I implacably insist that my students' draw pictures when they are presenting their work at the board. This takes patience and time, as learning to draw the pictures is a skill in itself. I try not to draw the diagrams for them, but early on I feel free to give lots of good advice to students who are stumped. Over time, my students come to expect to be asked to start by drawing a diagram. The best expositors learn to build diagram, piece by piece, as they explain their proof.

There are (at least) two distinct classes of useful diagrams. There are diagrams that illustrate what a theorem is saying and there are diagrams that illustrate the process that is used to prove it. Consider Theorem 3.1.12 which gives several characterizations of boundedness. The following diagrams illustrate what each of the statements in the theorem is saying.



It is possible to elaborate these diagrams to illustrate the proofs of each of the implications. I work with my students to learn how to make both kinds of diagrams. (I may or may not make the distinction explicit to them.)

Finally, I make frequent non-mathematical comments. I think our students benefit from comments that help them become better writers and speakers. In mathematics, correctness is always paramount, of course, but beyond this, good writing is about communication. How do we write correct proofs that help our readers the most? For instance, how can we

- with a careful choice of words
- by thoughtfully considering the order in which we bring things up

- by gentle emphasis of some things over others

write proofs that clarify, rather than obscure, our argument? Moreover, there are lots of writing conventions in the stylized "Kabuki dance" that is epsilonics. I believe it is important for us, as instructors, to gradually introduce our real analysis students to these elements of mathematical culture.

How do I decide when to assign class problems to specific groups of students and when to the whole class? Having individual students or groups of students working on and responsible for presenting specific problems can be a very efficient way of moving through material quickly. I don't recommend this for the day-to-day routine because there are many problems that all students need to be working on if important issues are not to go over their heads. However, some sections have lots of problems that all need to be solved. (Either because a subtle new concept takes some getting used to or simply because there are a lot of important results to establish.) It is frequently the case, for these sections, that several different problems use similar techniques or deal with the same ideas in slightly different ways. Students can learn a lot by dealing directly with the ideas and the techniques in one or two problems and seeing the way that other students dealt with them in slightly different contexts. This is an ideal section for having different students or groups of students working on different problems.

When I divide the class into groups, how do I choose the groups? I wish I had a magic method for doing this in the best possible way. Unfortunately, I don't. There are advantages and disadvantages to different schemes. I have done everything from just dividing up the class "geographically," based on the way that students are sitting in the classroom on the day when I assign groups, to trying to see which students already work together (or asking them) and assigning groups that respect that existing organic mix, to picking the groups myself and "rotating" throughout the semester so different students are working together at different times.

When I assign specific problems to groups or individuals, I often match the problems and the groups carefully. Students who are still struggling with basic definitions are given problems that they can make progress on, while the more difficult problems are assigned to students that are a bit more advanced and can handle the extra challenge.

What happens when no one has anything to present? This is a more difficult question to answer because the situation can arise for lots of different reasons. The extremes go from "it's midterm time and none of the students have prepared for class" to "the ideas or the problems are really tricky and, despite their best efforts, students just haven't made a breakthrough."[3] The distraction that comes from a lot of work in other classes is bound to happen at some point during the semester. I try to walk a tricky tightrope in this case. On the one hand, I think that it is unproductive to come down too heavily on the students

[3] It is usually easy to tell the difference between these.

for this. It is just a reality of college life to have times when there is too much work to get it all done. So I tend to be understanding, but I show the claws under the velvet glove. I gently point out that since real analysis was clearly the <u>last</u> priority in preparing for classes today, for the next class it had better be the <u>first</u> priority. Then I set the class working in small groups on what they were supposed to have prepared for class that day. (Chances are, the groups will start coming up with solutions, and we can stop other discussions to let them present their proofs. I often find that the class still gets through a lot of what I had planned for the day.) The students usually feel really badly and prepare well for the next class. However, if they don't, the class really feels my displeasure the next time. Guilt and shame are powerful incentives.

There are a few times during the semester that I fully expect the students to have to read, discuss, and then read again before proving any theorems. (For instance, when we first introduce compactness!) So I play a dirty trick and assign problems for, say, Wednesday, fully understanding that we won't be getting to them until Friday after the class teases out some nuances. For these days, I prepare helpful examples or exercises, discussion questions, etc. that can help the students make a breakthrough.

In other circumstances, conceptual issues are not the sticking point. The problems are just hard and students just need more time to have them come together. Having the students consider enlightening special cases, or suggesting a useful picture can help. I recently heard Ed Burger of Williams College say that he asks students directly: "what was the best idea you had that didn't work? Why didn't it work?" Then the class discusses the answer(s) that come up. (I haven't had a chance to try it yet, but I really like this idea and I can't wait for the fall to see how it works!) Some people I know have two or three cool topics stored away on which they can give an impromptu "big picture" lecture. This can be a nice side-trip and can be provide some enticing mathematical enrichment while giving the students more time to think about the problems.

What do I do with a very shy students that really *don't want to present a proof in front of others?* Mostly, I gently encourage them in class. If they don't react well to this, I back off and take my encouragement outside of class. Usually, reluctance to present is a combination of shyness and lack of confidence. If I see this in a student, I may privately offer to let him or her practice a presentation for me, in my office, before presenting the proof to the class. This boosts the student's confidence and makes it much easier for him/her to get up in front of peers to discuss mathematics. In all my years of teaching, I have only once had a *really* good student[4] who was so painfully shy that it was almost impossible to get her to the board. I offered to let her make some private presentations to me before proofs were presented in class—which was itself a huge step for her—and counted them as though they were presentations made to the class. But this was a really exceptional situation.

[4]Made darn near 100% on the takehome exams—I am pretty sure she had most class problems solved before they were presented by others.

I think that, unfortunately, giving a lot of credit for class presentations does favor the extraverts. However, all students seem to grow more comfortable with presentations over time. And I know that many of our graduates say that making presentations before a small group is a frequent occurrence in their jobs. They thank our department for making this a standard part of the curriculum. Thus I think that giving introverted/quiet/shy students a chance to grow into a "public persona" is, in fact, really good for them.

Who decides when a proof is correct? We want to encourage our students to develop their own sense of what constitutes a correct proof so, ideally, the students in the class will work to find a consensus about whether something is correct or not. But, in order to inculcate this sense, we have to convey to our students, professional standards about what it means to prove something. It is important for them to understand that a string of true statements that starts with the hypothesis and ends with the conclusion is not automatically a proof. The statements must be linked by logic. If Chris says that such and such (a true statement) implies so and so (another true statement), we have every right to ask "why?" Because we are aiming for a long (mostly) unbroken chain of logical reasoning, in my class the only acceptable justifications are: "by axiom thus and so" or "by theorem such and such" or "by the definition of whooziwhatsit." Other important questions for our students are: "is it clear?" and "is it complete?"

What happens if there is a mistake in a problem and the students don't see it? I don't let it go. I start quizzing the students in the class about various portions of the problem, using the standard questions mentioned above. I usually doesn't take long for someone to see the problem.

When is "clear" really clear*?* Suppose Alex says:

> Let $r > 0$. It is clear that $U = \bigcup_{a \in U} \mathrm{B}_r(a)$.

Do we accept this statement? Well, there obviously comes a time when we do. But probably not the first time (maybe not the second or third time) it comes up. In order to get my students used to making precise statements about balls, distances, and so forth, I would initially expect them to explicitly write out the (admittedly pretty trivial) element arguments necessary prove this. Because they are trivial, we might talk about why the statement is true and try to nail the justification in only a sentence or two. (Being concise and also precise is a skill to work on.) Later in the semester, when this sort of statement has become routine, I would loosen up and let Alex assert it. However, if challenged, Alex must always be prepared to justify any statement she makes by using the standards of proof set by the class.

Testing and Grades

When I teach Real Analysis, and I divide up the grade as follows:

Class participation and in-class presentations	25 %
Written assignments	20 %
Quizzes	5 %
In-class Midterm	5 %
Take-home Midterm	20 %
In-class Final	5 %
Take-home Final Exam	20 %
Total	100 %

A large portion of the grade is given for the work that students do day to day for and in the class. All students are expected to participate regularly in class discussion and to put in their fair share of time at the board. Since students cannot participate if they are not present, class attendance is mandatory. I don't make the "teeth" on this policy explicit, but I keep track and missing classes without good reason is understood to count against the student's participation grade at the end of the semester.

I am often asked how I evaluate student presentations. Over the years, I have tried many schemes for evaluating student presentations.[5] In the end, of course, my goal is for students to be able to prove theorems. I measure students ability to do this with takehome exams on which they are asked to prove theorems they have not seen before. My experience has been that, in the end, the thing that correlates most strongly with good performance on the takehome exams is, plainly and simply, the number of times a student goes to the board to present his or her work. Thus over the last several years, all I have done is to keep track of which problems are presented by which students. I use the number of times that a student has presented his or her work to compute the grade for this portion of the course. I am thinking of adding a "distribution" requirement as there are some students who madly start trying to present toward the end of the semester in order to salvage this portion of their grade, which is not nearly as valuable as working regularly on class problems throughout the semester.[6]

As you see, I give two exams during the semester. Each of these exams has both an in-class and a take-home portion. The in-class portion is worth much less than the take-home portion.

[5] See the box at the end of this section for details

[6] An incorrect presentation never counts against a student in my course. Indeed, a student who gives an incorrect presentation is given the chance to correct the proof—with my help, if necessary—and present a correct proof the next class period. Sometimes I give the student the option of passing; if the student chooses to cede the presentation to someone else in the class, there is simply no record of the incorrect attempt. I find that this nothing-to-lose approach encourages some weaker or more reluctant students to volunteer. And I think that is a good thing.

The in-class portion is meant to be a straightforward, objective test that measures how well students have the "facts" at their fingertips. I always ask the students to give several definitions. The rest of the test consists of true/false or short answer questions. These often (but not always) require a short justification. Among the short answer questions I usually include some that ask the students to give examples and counterexamples of objects we have studied. They are also responsible for knowing the statements and significance of major theorems. If the students have kept up with the readings and discussions, putting the ideas together in their minds, they should be able to do well on this portion of the exam.

I do not ask the students to prove theorems on my in-class exams. Instead, I give a few quizzes (5 or 6 during the semester) in which students are asked to write the proof of a theorem that is proved in the book. I started doing this when I realized that students were not reading the proofs with any attention because I was not holding them responsible for doing so. My students know ahead of time which theorem or theorems they will be responsible for on a quiz.

The take-home portion of the exam consists entirely of proofs that the students haven't seen before—usually 6-8 problems, several of which have multiple parts. I include a wide range of difficulty in these proofs. I give a couple of easy and short arguments that just test the students' ability to "follow their noses" through the logic from the definitions to the desired conclusion. Each exam also has a more difficult proof that requires a real idea or a deeper understanding of ideas covered in the text (or both). Certainly such a proof will have several steps so that the students will have to sustain a chain of reasoning in order to achieve their goal. Most of the problems lie somewhere in between. In addition, I have some general sorts of things that I like to include on exams.

- I often include a problem in which students are asked to decide whether a certain mathematical statement is true and to justify their conclusion by giving a proof or a counterexample.
- I also like to introduce a new idea by giving the students a definition and asking them to prove some some simple propositions about the new idea.
- Many times I turn a harder problem into one of medium difficulty by breaking it up into multiple parts that help the students find their way through the ideas and also give them a good chance for partial credit.

Time frame for takehome exams: Students must balance the competing demands made by their various classes. I used to give my students a week to work on their take-home exam, thinking that it would not be so hard for them to find a couple of days within that period to dedicate to the work. However, I discovered that some students were spending most of the week working on it, while others (who were unlucky enough to have other papers due or other tests to study for) could only use a day or two of the time. I tried making it just a two-day turnover, but then some unlucky students got to spend very little time on it. It never seemed to work out fairly. I have at last arrived at a compromise scheme. I seal each test in a manila envelope. Each student can then pick any 48 hour time period during a specific one-week span to work on the exam. The students are on their honor to work only for the 48 hours. I have more recently started asking my students to LaTeX their exams. Since I have done this, I give them an extra 12 hours—for a total of 60 hours—to work on the exam.

Other methods I have used for evaluating student presentations:

Very Simple Student gets a score of 1 for a proof that was eventually correct but in which corrections were made during the presentation and a score of 2 for a presentation that is correct on the first time through.

Pretty simple Each presentation gets a score of between 1 and 10. I take into consideration how well the student communicates and whether the problem is correct or whether minor corrections are made in the course of the discussion.

A bit more intricate Each presentation gets two scores each of between 1 and 10. The first score is a presentation score—how clearly written the proof is and how clearly the student has communicated the important transitions, etc., in the proof. The second score is based on the mathematical correctness of the final solution. It also takes into consideration whether minor corrections and amendments are made in the course of the discussion.

Most intricate Each student presentation had an evaluation from me and separate evaluations from two of his/her fellow students. At first I didn't evaluate the student evaluations. This made them more or less useless because the student evaluators were reluctant to be critical of their classmate. When I started evaluating the student evaluations before passing them off to the student who was evaluated, the quality of the evaluations increased, and the exercise became useful for both the students doing the evaluation and the student being evaluated. This had some educational value, but it was a logistical nightmare for me and for the students, and (in the end) I did not feel that the benefit was worth the effort.

Part III
The Book—in detail

General Remarks

Here is a list of miscellaneous remarks that I think may be useful to you. They are in no particular order.

- I recommend the *Note to the Student* in the front matter of the book for teachers as well as for the students who use the book.

- "Code words" in the book—generally speaking, problems or exercises that say "give an argument" or "explain why" or "give an example to show" are meant to help illustrate or clarify a specific point. They are not part of a larger assumed chain of reasoning in the book. Thus I often don't ask my students to give a careful proof. A looser argument that might be turned into a proof is best, but a nice discussion is usually OK, too.

- I have found that it is a good idea to encourage students to use $|x - y|$ rather than $d(x, y)$ when working in the reals. There is obviously no mathematical reason for this. But I find that it makes students pay specific attention to the context within which they are working. This deliberate focus, mentioned non-chalantly but explicitly in class, seems to translate into fewer students using the absolute value distance in contexts where it isn't warranted.

- I like to stress to my students that, from the point of view of mathematics, equivalent conditions are just different ways of saying the same thing. To help reinforce this idea, I may introduce a new idea with a definition and then give a theorem that shows several equivalent conditions. Or, I may give the theorem first and define the term afterwards by saying that it is that thing which satisfies any one of the various equivalent conditions given in the theorem. Once we have the equivalences, we use the conditions interchangeably without comment. When I ask my students to define a term on a test, they know that any one of the several equivalent conditions is fine.

- I believe that "trivial" results with "easy" proofs can be pedagogically important. Even if the students themselves see these as extremely straightforward 48 hours after they first encounter them, they can be very good for helping the students work their way through the details of a new definition. Or they can make it easy to discuss the structure of a certain sort of proof. Or they can focus the students' attention on specific aspects of a new concept. So most sections in the book have one or two of these, either as exercises in the body of the section or as early problems at the end. Whenever possible, I tried to have them take the form of straightforward principles that can be useful as little lemmas or techniques at other points in the book. (Examples: problem 1 at the end of Section 3.3, problem 1 at the end of Section 3.7, problem 2 at the end of Section 4.3, exercises 7.1.4 and 7.1.5.)

- There are problem "themes" in the book: sets of several problems spread over different sections that all revolve around a single theme or idea. Some examples are:
  - **isolated points and discrete spaces**—problem 8 at the end of Section 3.1, problem 5 at the end of Section 3.3, problem 2 at the end of Section 3.5, problem 9 at the end of Section 3.6, problem 8 at the end of Section 4.3.
  - **the metric space** ℓ_∞—problem 9 at the end of Section 2.2, problem 2 at the end of Section 3.1, problem 5 at the end of Section 6.2, problem 14 at the end of Section 7.1.

Chapter by Chapter

In the following pages, I make brief remarks about each chapter and excursion. These are meant to give you an overview of the sections, from my point of view as a teacher. My remarks are based on my own perspective and are not meant to be taken as some sort of final or even definitive way of thinking about the material. They certainly do not rise *close* to the level of "instructions" for teaching out of the book. Along with my brief comments I try give you a "birds-eye" view of the problems at the end of the section. This is particularly useful, because some results that appear only as problems are, in and of themselves, of peripheral interest. But they are useful tools for proving other more interesting results that show up later on in the book. I try, whenever possible to give you a "heads-up" about this.

When I say that I think of a problem as "essential," you should understand that to mean that the result is central to the theory and probably also that it will be needed for proving theorems that appear in subsequent sections and chapters. Occasionally I will list other problems as "also very useful," or something similar. This should be taken to mean that the result will prove to be useful later in the book. Tread a bit lightly in choosing not to assign these, or be prepared to double back and assign them later on when they become needed. I highlight problems are good for helping students understand an important idea. When an innocent-looking theorem is surprisingly difficult to prove, I try to warn you about that. There will be some problems that I don't mention at all. You should not take this to mean that I don't think they are worth considering. I think they have value and interest or I wouldn't have put them in the book. But if I don't mention them, certainly it is "safe" to skip them without concern that it will come back to haunt you later.

Preliminary Remarks

None of this is really crucial for what comes afterward, but the two essays *What is Analysis?* and *The Role of Abstraction* are short and easy to read and make a nice way to "set the stage" for what is to come. The *Thought Experiment* is a good exercise for the first day of classes. I have my students work in small groups on this "experiment" and we discuss it only as things come up while they are working on it during that first class period. In my mind, the purpose of the exercise is to make my students aware that they do not come to the class unprepared. The intuition they built up in their calculus courses is useful and they should be sure to bring it along on their journey. At the same time, it is insufficient even for proving statements that are "obvious" to them, much less for moving them beyond their prior knowledge. I also like to have this as a first day activity because it gets students actively engaged from the very first moments of the course.

Chapter 0—Basic Building Blocks

One presumes that much of the information on sets, functions, and mathematical induction will be familiar to students who are taking this course. If your students are not familiar with these ideas, it is worth spending some time on things like showing sets are equal, one-to-one and onto functions, images and inverse images, and so forth.

Even students who have seen earlier information in great detail (as mine have) will likely be unfamiliar—or only marginally familiar—with the information on sequences and subsequences that is treated in Section 0.4. This information is very important, as it crops up again and again in the study of real analysis. I, myself, do not cover Section 0.4 at the beginning of the course. Instead, I wait until I am about to cover sequence convergence in Chapter 3 and double back at that point.

Theorem 0.4.6 and problem 4 are very useful results. The theorem that every sequence in a totally ordered set has a monotonic subsequence (problem 5) is extremely important and will also crop up later on. Unfortunately, because this is a deep theorem, its proof is quite tricky. A few of my best students are able to handle it and thrive on the challenge. Most really struggle and, in the end, don't quite "get it." There is some value in this struggle, but it takes time that I might rather spend on other things. So I go back and forth about whether to assign the problem to everyone. It is good to entertain various options. For instance, one could put an exceptional student or two in charge of working on the problem and presenting it to the class. Alternatively one could discuss the general idea behind the result in the context of $\mathbb{R}$ and convince the students it's true. (An arbitrary enumeration of the rational numbers is a good example to consider.) Then the class can just agree to use the result where needed, but leave the proof "hanging" as a challenge problem.

Regarding the construction of sequences and subsequences: It is worth considering, up front, whether you want your students to produce formal induction proofs when they construct sequences and subsequences. The alternative is for them to explain how they will pick the first few terms and to do so sufficiently well that it is clear the process can be sustained and that the sequence satisfies the desired condition(s). Formalizing this process with induction is hard for most students; it takes time to train them in the proper construction of these arguments. If you are happy with the more informal approach, it will definitely save time that can then be dedicated to other things. I, myself, am ambivalent.

Chapter 1—The Real Numbers

I begin my course with Chapter 1. The birds-eye view is that Sections 1.2 and 1.3, while important, are not really analysis. After my students leave Chapter 1, I allow them to assume the standard results of elementary algebra, including results about inequalities. We prove the most important of them at this point in order to see some of the basic connections, but we don't subsequently dwell on them. In view of this fact, the most important theorem for subsequent study is Theorem 1.3.8. Most students' grasp on the algebra of absolute values is pretty tenuous, and few think of absolute values in terms of distances on the real line. Part 8 is especially helpful for beginning to build some insight, both about measuring distances and about the relationship between analytical ideas and geometric ideas. (This is an opportunity for some helpful pictures, as well.)

There are a lot of results in Sections 1.2 and 1.3 and it is easy to get bogged down. Because there are so many little theorems and their proofs are so similar in flavor, I have taken to breaking the class into small groups and assigning each group 2-3 "mini-results" which can then be presented to the class very efficiently. This has the advantage of getting absolutely everyone to the board in the first week of classes.

The first real look at "closeness" comes with the least upper bound property. The proof of Theorem 1.4.4 (equivalent conditions for the least upper bound) is worth a bit of discussion, as it is the first analytically flavored argument. Moreover, the result frequently comes into play in later sections. Problems 2 and 8 at the end of Section 1.4 are also nudges in this direction and are important later on. The proof that every positive real number has a square root (Theorem 1.4.7) is fairly hard slogging for students at this point, and I have stopped spending class time on it. I note that it is there if students want to work through it and offer to help outside of class if they have questions.

Chapter 2—Measuring Distances

In this chapter one wants to get the idea of distance on the table and to prove the basic facts about distances in $\mathbb{R}$ and $\mathbb{R}^n$. Furthermore, problems 1-3 at the end of Section 2.2 are extremely useful results, and I recommend them. I find that a short discussion of the result in problem 1 and its usefulness is helpful. The proof is easy, so the significance of the result can easily go over students' heads. The fact that, with limiting processes, approximate equality is often easier to prove than exact equality makes this a very useful tool to put in one's bag of tricks. Though the proof doesn't require it, a picture can be very helpful for illustrating the result in problem 3.

Less crucial for future work, problem 5 gives students a workout on the properties of a metric and is a good, straightforward problem to assign. Problem 9 gives students an example of an easy-to-picture infinite dimensional space and is also a good problem. There are several problems, spread out throughout the core chapters, that build on problem 9 by considering various properties of the space ℓ_∞.

Chapter 3—Sets and Limits

Section 3.1—Open Sets

It is easy to underestimate the difficulty of this section. The arguments are easy, but the material presents some conceptual hurdles for the students. The section introduces three concepts of (perhaps) unexpected subtlety: open balls, open sets, and boundedness in metric spaces. The students must get their heads around the definitions—what they mean, how to think about the concepts. But there is also the more difficult task of mastering how to use the definitions in proving theorems. As mathematicians, we have developed ingrained reflexes about how to use new definitions in a proof, but this is not yet natural for most students. They will be distracted by their intuitive understanding of the concepts. As teachers, we have to work to focus their attention on the rigorous formulations embodied in the definitions. My experience says that with these new concepts we will need to do this repeatedly before it really sinks in.

Because the concepts are vitally important for everything that comes afterward, it is worth spending some time here. There are a lot of good problems at the end of the section that can be used to hone the students' understanding of the material.

- Problems 1 and 10[7] are absolutely crucial for future work and should definitely be on the agenda.
- Problems 3 and 4 are also useful facts. As a bonus, they are easy results in which students must work with the definitions of open set and open ball. I always assign them.
- Problems 2, 7, and 8 are not especially crucial for future work, but they are all excellent for getting students to move beyond their naive conceptions of open ball and open set. (They help students see what the definitions *do* say and what they *don't* say.)

[7] Regarding 10(b), it is a good idea to point out to students that in part (a) they prove several equivalent conditions for boundedness. If they think carefully about which to exploit, 10(b) is a corollary to part (a). Otherwise, they will end up working much harder than they need to. Part of the idea in nudging them explicitly here is that the importance of part (b) goes beyond just establishing the truth of Corollary 3.1.14. There is something important to be learned about boundedness and how to exploit it in proofs.

- Problem 11 is a good workout with the concept of boundedness, but it can be skipped. There will be problems later on in which the various characterizations of boundedness can be reinforced.
- In problems 6 and 12, the students must "get their hands dirty" with the structure of $\mathbb{R}^n$ and this is always difficult. Students are very reluctant to buckle down and define coordinates for elements of $\mathbb{R}^n$; the notation is cumbersome, so they try to avoid it. But it is absolutely essential in problems such as these.
- Problem 13 is really challenging.

On problems involving $\mathbb{R}^n$: I used to skip most theorems in which the linear structure of $\mathbb{R}^n$ plays a central role. Students hate them and the results don't usually come into play until much later on—for instance, in technical results about the calculus of several variables. After all, one of the advantages to doing things in a general metric space is that you can, mostly, avoid the cumbersome n-tuple notation and the ugly distance formula. However, I have more recently come to the conclusion that it is important to train our students to just deal with nasty notational issues when necessary. In the worst case scenario, their reaction to such problems should be to be annoyed and/or bored by the notation and the algebra, not petrified into inaction. The only way to get our students to this point is to make them do problems. Moreover, if we are serious about showing our students how the theory plays itself out in concrete spaces such as $\mathbb{R}^n$, then we have to assign them problems in which the structure of $\mathbb{R}^n$ plays a crucial part. But it is definitely a judgement call. It is perfectly reasonable to "store up" some of these problems and to have students prove them as an prelude to studying, say, the calculus of several variables.

Sections 3.2 and 3.3—Convergence of Sequences

Section 3.3 gives us our first real limiting process. In definition 3.3.1 students see, for the first time, the stacked quantifiers that characterize analytical definitions, and it leaves them completely stymied. Exercise 3.3.6 is an excellent way to get students to really *engage* the definition of sequence convergence in a practical way, but they aren't likely to do this on their own. (The exercise looks to them pretty much like the definition of which they could make neither heads nor tails, so they shy away from it.) On our first day on this topic, I divide my students up into groups of 4 or 5 students and ask them to work through the exercise. In a group they make good headway, and they to begin to see how quantifiers make the definition work. I usually assign problem 1 in Section 3.3 and problem 2 in Section 3.4 as "warm-ups" for the definition.[8]

[8] Problem 2 in 3.4 requires NO information beyond definition 3.3.1. It is a simple convergence result from the real numbers and the only reason it is in Section 3.4 is that it is a real number result.

These are both useful little results and are easy applications of the definition. I can usually get the groups through these two problems by the end of that first class period. As they work on these, the class as a whole talks about how to phrase arguments that use definition 3.1.1, because langauge tends to be an initial stumbling block, as well. After this, progress better, though students are still struggling with the new way of thinking about the world.

Problem 2 (uniqueness of limits), problem 4 (boundedness of convergent sequences), and problem 6 (subsequences and convergence) are all very important results and should probably be assigned. Problems 3 and 5 are good workouts on the definition. Problems 7 and 8 are somewhat more difficult and are useful results, though certainly less crucial than 2, 4, and 6.

Section 3.4—Sequences in $\mathbb{R}$

In this section, students see how order and arithmetic in $\mathbb{R}$ interact with sequence convergence. The results are "segregated" in a section of their own to emphasize that the results don't apply in the more general, abstract context that students have been studying thus far. Because of the more elaborate structures that are in play, specific computational issues arise that have not yet been encountered. (These are exquisitely illustrated by the proof that the limit of the product of two convergent sequences is convergent. But there are also issues that arise in connection with the ordering on $\mathbb{R}$.) Theorem 3.4.9 is without a doubt one of the most important theoretical results in the section and should certainly be proved. At least a couple of the other theorems involving inequalities and Theorem 3.4.11 are, from my point of view, also essential results. I usually combine study of Section 3.4 with study of Excursion D in which some similar issues arise.

Sections 3.5 and 3.6—Limit Points and Closed Sets

Sections 3.5 and 3.6 introduce crucial ideas and make some useful connections, but the problems are very easy and the techniques are beginning to be familiar by this point, so it makes sense to think about going through these sections together and fairly quickly.

In Section 3.5, Problem 1 makes some useful connections. None of the rest of the problems in this section are crucial, at this point, but δ-separated and dense are nice concepts to introduce at some point and problems 3 and 4 give good workouts with the definition of limit point. In Section 3.6, problem 3 is crucial. Problems 1, 2 and 5 give nice workouts with the definitions. Problems 4 and 6 do, as well, but are a tiny bit more difficult. Others are nice problems, but optional. Note: Problem 10 is fairly difficult for students at this stage, unless problem 13 at the end of Section 3.1 is assumed.

Section 3.7—Open Sets, Closed Sets, and the Closure of a Set

The most crucial result in this section is Theorem 3.7.1. Closure, interior, and boundary are all useful concepts, but can be de-emphasized if time is short.

I usually cover the closure, but often leave the other concepts for good takehome exam problems.

Chapter 4—Continuity

Section 4.2—Limit of a Function at a Point

This second definition of a limiting process goes down much more smoothly than the first one. There are some new wrinkles, but the general process is now familiar. Nevertheless, there are some important nuances. A bit of class discussion on the question to ponder at the top of page 108 is very fruitful. Exercise 4.2.2 and problem 1 can also make for some nice class discussion that works with the definition. The only really crucial problem (from the point of view of future studies) is problem 2 in which students prove that limits are unique. It is worth making sure students see that the requirement that a be a limit point of the domain is really crucial in this argument. Bringing in at least some of Excursion E can also be useful at this point.

Section 4.3—Continuous Functions

A lot of things come together in this section. Problems 1 and 4 are, of course, crucial theorems. Students that really take to heart the three equivalent conditions for continuity given in Theorem 4.3.3 can make their lives very easy in problem 4. Otherwise, they will end up working much harder than necessary. This is a good lesson. (It may be worth asking for alternative approaches to this problem so that students really see this principle at work.) In addition, I like to assign problem 2, which is a surprisingly useful little result. In addition, its proof is *really easy* if students only understand the definition of continuity and can apply it. But it is not possible to do this on some sort of "autopilot." Problem 5 is a nice version of the inverse function theorem and is a bit more difficult but not extremely so. There are some other interesting problems from which you can choose, but they are optional.

Section 4.4—Uniform Continuity

This section will prove a bit more subtle than the previous because students will have trouble, at first, distinguishing between continuity and uniform continuity. A general "heuristic" discussion is helpful even before the reading is assigned. Problem 1 is somewhat tricky, but is also very helpful for teasing out the issues. Problem 2 is important enough that it should probably be assigned. Lipschitz functions play a large role in the theory of iteration, so you should definitely assign problem 2 if you plan to cover Chapter 10. Problem 3 is also good because it gives a viable (in the sense of proving theorems) characterization for what it means to fail to be uniformly continuous. It will prove useful in later sections. Problem 4 is easy and useful, too. Problem 5 is a bit more difficult, but is also a useful result.

Chapter 5—Real-Valued Functions

In terms of content, this chapter is much like the section on sequences of real numbers (Section 3.4). From a pedagogical point of view, it is a good time to emphasize the connection between sequence convergence and limits/continuity. Because quite a few of the theorems can be gotten, more or less, for free from their sequence counterparts, I find that students pay attention (more or less for the first time) to Theorem 4.2.4.

Chapter 6—Completeness

Certainly a very important concept. At the very least the idea of a Cauchy sequence and the completeness of $\mathbb{R}^n$ needs to be covered. (This will play a crucial role in the Heine-Borel theorem.) If time is at a premium (and it always seems to be) I sometimes cut corners and do just this much. But if there is more time, problem 3 at the end of Section 6.1 is an extremely useful little theorem. Moreover, the result that every closed subspace of a complete space is complete (problem 3 at the end of Section 6.2) comes in very handy from time to time. Also, from a pedagogical standpoint, the proofs of both these theorems bring earlier concepts together in interesting ways.

Chapter 7—Compactness

There are three big focal points in this chapter. In Section 7.1 students learn about the nature of compact sets. In Section 7.2 students begin to get some idea why this is all worth it, when they see that compactness is the key to the max-min theorem. In Section 7.3 students fully understand the nature of compact sets in $\mathbb{R}^n$.

Section 7.1—Compact sets

Because compactness is a completely new idea (and kind of a bizarre one, at that), we work a fair number of problems in this section. Certainly my students prove Theorems 7.1.7, 7.1.9, 7.1.10 and ($2 \Longrightarrow 3$) in 7.1.11. I usually do not take the time to make them struggle through ($3 \Longrightarrow 1$) in Theorem 7.1.11. This is quite a difficult theorem and (even with hints) most of my students would find it overwhelming.[9] It might be a great challenge for an especially talented and ambitious student. Of course Theorem 7.1.12 is also an important result and very easy, in view of Theorem 7.1.11.

Problem 11 at the end of Section 7.1 is a technical lemma, but a very useful fact that comes in handy more than once for important theorems in the section on continuity and compactness. I think that problem 14 is absolutely crucial for understanding that compactness is not the same as closed and boundedness. This gives the students some idea what the fuss is all about when we go to a lot

[9] I usually tell my students, outright, that we are skipping this proof because it is really hard. They find this to be a cheerful thought, so there is a nice psychological benefit. (I'm giving them a break, for once!) But I do it for another reason. I think it is important for them to know that that this is a very deep theorem. I reinforce this fact when I discuss the Heine-Borel theorem.

of trouble to prove the Heine-Borel theorem. Problem 15, the generalization of the nested interval theorem, is also a very important and useful theorem. One cannot, of course, do everything. Sometimes I deliberately leave one of these results for the takehome final.

Section 7.2—Continuity and Compactness

The focal point of this section is, of course, the max-min theorem, but working with compact domains and continuous functions is also incredibly important. My students always work problems 1, 2, 3, 4, and 5, some in class, some for homework. It is this section that begins to give the students a clue as to why anyone would care about a weird notion like compactness.

Section 7.3—Compactness in $\mathbb{R}^n$

Despite the fact that the Heine-Borel is proved in great detail in the section, I usually take a day to lecture on the proof. I really want my students get the big picture of the argument and if I just leave them to read the proof, I think most will be mired in details. This is the only full-blown lecture I give when I teach the first semester of real analysis. (The lecturing also works well with the pacing of the course. My students' out-of-class time is taken up by a "killer" homework assignment from the previous two sections.)

I hand out a set of "lecture notes," which basically consist of the information that is already in the book with spaces left for additional note-taking. This allows my students to pay closer attention to what I am saying and to take notes on things that need clarification. Moreover, it prevents me from having to write everything that is in the book on the board. If you would like a copy of this handout, please feel free to contact me, I would be happy to send you one.

There is an important thing to note for the students. The proof of the Heine-Borel theorem would be greatly simplified if one were simply to prove that every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence. But this simplification is really illusory because in order for this to yield a rigorous proof of the theorem, one would need to proof that 3 implies 1 in Theorem 7.1.11. And this is even harder than the proof given for the Heine-Borel Theorem.

Chapter 8—Connectedness

From the point of view of future work, the "plain vanilla" intermediate value theorem (Theorem 8.1.2) is the most important thing in this section. This can "be proved directly by appealing to properties of the real numbers," so no additional background is really needed. However, from a curricular stand-point, I think it is worth making sure the students know that the important issue in the IVT is the fact that the domain is connected—not that it is compact or some other property. So having students at least read the introductory material in Section 8.1 is probably good. Problem 2 at the end of Section 8.1 is needed for the proof of Darboux's theorem in the chapter on differentiability. Problem 3 is a version of the inverse function theorem and is also, therefore, interesting.

The general discussion of connectedness in Section 8.2 has more general applicability, but it can be skipped if time is short.

Chapter 9—Differentiation: One Real Variable

Section 9.2—The Derivative

From a pedagogical standpoint, the main thrust of this section is to acquaint students with the local linear approximation version of the definition of differentiability—what it is, what it means, how it is related to the more familiar difference quotient formulation, and how to use it. But the section also establishes some of the basic properties of differentiable functions. The continuity of differentiable functions and the uniqueness of the derivative are established. Moreover, we get the arithmetic differentiation rules and the chain rule. Problems 3, 4, 5, 8, and 11 seem absolutely essential to me. Problem 6, which establishes the uniqueness of the derivative, is also important. (Though it may seem less important to our students than it does to us!) The existence of a differentiable function with a non-continuous derivative is highly non-obvious and is an important thing to establish, so problem 12 is also recommended.[10] Problems 9 and 10 establish the quotient rule. I may or may not assign these to my own students, depending on the available time. There are some other interesting problems that can be used to sharpen the students' intuition about differentiability but each is optional, in my opinion.

Sections 9.3 and 9.4—The Mean Value Theorem

The mean value theorem is, of course, the most important theorem in the chapter. Though most students will have seen it in a calculus course, they are almost certainly unaware of its true significance in the theory of differentiable functions. In Section 9.3 I attempt to establish the need for the mean value theorem. The theorem is proved in Section 9.4, as are a couple of the most important corollaries. Problems 1, 2, and 3 are absolutely essential, as these constitute the proof of the mean value theorem. Problems 4 and 5 establish the standard corollaries about constant functions and antiderivative families. In proving these, the students see how the mean value theorem is used to obtain theoretical results. Problem 9, on Lipschitz functions and differentiability, also serve this purpose and make some important connections. Problem 8 is the basis for establishing L'hôpital's rule (not in the book, I'm sorry to say!), if that interests you.

Section 9.5—Monotonicity and the Mean Value Theorem

Clearly one wants to establish the relationship between monotonicity and the derivative, but this section also establishes Darboux's theorem, which delves much more deeply into the relationship between the behavior of the function and the behavior of the derivative. In my opinion, this is the most interesting theorem in the chapter, for two reasons. First it is a beautiful and surprising result,

[10] It is interesting to have students try the "obvious" ploy of starting with a function with a jump discontinuity and then computing an area accumulation function. When they become convinced this won't work, problem 12 and, later, Darboux's theorem become a lot more interesting.

something that students don't expect to see in the chapter on the derivative. It shows them that a deep look at the theory underlying differentiation really does tell us things that we didn't already know. This is a valuable lesson, especially for those students who feel that proving theorems amounts to nitpicking and nothing more. To get all of this, you will need to assign problems 1, 3, 4 and 5. Problem 6 is also helpful for making students think about the significance of Darboux's theorem.

Sections 9.6 and 9.7

These sections on inverse functions and Taylor polynomials give important culminating results and are important in their own right. They can be skipped if time is short; however, you will need Taylor's theorem if you plan to cover the section on Taylor series in the excursion on power series.

Chapter 10—Iteration

Iteration is an important theoretical tool in analysis. Until fairly recently, it has been thought of *purely* in terms of its theoretical significance for very high powered theorems. It didn't, therefore, tend to be a focus of introductory courses. With the advent of the personal computer, iteration has taken on a much more visible role in both pure and applied mathematics and I believe that it now makes sense to have it play a more central role in introductory analysis. *Closer and Closer* reflects that perspective.

Section 10.1—Iteration and Fixed Points

This introduces the idea of fixed points and establishes their connection to iteration. Theorem 10.1.7 is the only crucial result for moving forward; its proof is requested in problem 6. But I believe that, from a pedagogical standpoint, some additional work with iterated functions is very helpful. Some preliminary experimentation with concrete functions along the lines of Exercise 10.1.3 and problems 1 and 2 is very good for helping the students' intuition catch up with the ideas. (I reserve a computer classroom for a day, but this could also be assigned as out of class work.) A bit of "off to the side" theory such as that in problems 4 and 5 is also helpful for learning to handle the definitions. The information on attracting and repelling fixed points is central to the iteration of real-valued functions, but it can be by-passed if you are covering iteration without first covering the chapter on differentiation. However, if you plan to cover the excursion on Newton's method, it is absolutely essential. If you do cover attractors and repellors, problem 10 is the main theorem. Problems 8 and 9 are good for putting some examples into the students heads and giving them some perspective. Problem 11 is sort of a generalization. If your students work with period doubling in problem 12 (tricky!), a 20 minute discussion of some beautiful related ideas (e.g. Sarkovskii's theorem, the Feigenbaum fractal) can really get students excited about iteration.

Section 10.2—The Contraction Mapping Theorem

The contraction mapping theorem is the most important theorem in this chapter. Despite the fact that its proof is not particularly difficult, it is a remarkably deep result with far-reaching consequences. It's proof is outlined in problem 5. Problems 1-4 give some insight into the nature of contractions. Problem 4 is the most important of these.

Theorem 10.2.6 is a generalization that is sometimes easier to apply than the CMT, itself. Its proof is outlined in problem 7. Theorem 10.2.7 is an interesting modification, but is not as useful. A detailed outline of the proof is given in problem 9, which brings a fairly difficult proof down to something of only moderate difficulty. Both of these theorems are optional, if you are pressed for time. However, be aware that Theorem 10.2.6 is used in the excursion on solutions to differential equations. (Excursion O).

Section 10.1—More on Finding Attracting Fixed Points

The contraction mapping theorem is very important, but it is sometimes difficult to use because the conditions don't apply globally. Theorem 10.3.3 and Corollary 10.3.4 are local theorems that move beyond the contraction mapping theorem. They play a key role in the proof of the implicit function theorem (Excursion M). From a pedagogical standpoint, their proofs bring together a number of different techniques and ideas and are worthwhile for this purpose. Nevertheless, if you don't plan to cover the excursion on the implicit function theorem, this section is completely optional.

Chapter 11—The Riemann Integral

Section 11.2—The Riemann Integral

One thing that sets this chapter apart from earlier chapters is the cumbersome notation that is necessary for the analysis of the integral. Some students will resist the need to define and use the notation, looking for ways to get around it. We all know this doesn't work! Forearmed with this knowledge we, their teachers, can take some early steps to head off this avoidance behavior. Somewhat computational problems such as 2, 3, 4, and 5 at the end of Section 11.2 are not particularly important results in their own right, but they do give the students an early workout on the definition and get them using the notation right from the start. Problems 1 (uniqueness of the integral), 3 (existence of a non-Riemann integrable function) and 7 (Riemann integrable functions are bounded) are extremely important results and should certainly be assigned. Only the proof of that Riemann integrable functions are bounded is challenging and it comes with a suggested outline. Problems 8 and 9 are nice, as well, since students must bring to bear their geometric intuition about integrals to help them see how to proof the theorems. They are, however, completely optional from the point of view of moving forward.

Section 11.3—Arithmetic, Order and the Integral

The standard theorems about the properties of the integral are very important for moving forward, as they will be used repeatedly in the deeper analysis of the integral. Thus problems 1 and 4 need to be assigned. Problems 5 and 6 are also important theorems requiring meaty proofs of medium difficulty. (Each of these have several parts and I find it convenient to divide my class into small groups, with each group responsible for proving and presenting one or two of the parts.)

Section 11.4—Families of Riemann Sums

The section begins by discussing the family of Riemann Sums formed by all the Riemann sums on a given partition. It introduces the upper and lower sums as upper and lower bounds for this family. The discussion up through Theorem 11.4.6 on page 222 establishes some connections and teases out some counter-intuitive nuances in the theory. All of the results in this portion are fairly straightforward and can easily be proved by your students. This all leads up to a proof of the Cauchy criterion for the existence of the integral, which is the most difficult technical result in the section. Proofs of (technical) Lemma 11.4.8 and Theorem 11.4.9 are written out in detail in the text. Despite the full glory of detail, these results are quite tricky and students will find them tough slogging. Even if they can convince themselves of the details, they are likely to lose the intuition that underlies the technical points. Thus, I always lecture on these two results. (Because I have some discretion about the order in which I cover topics, I have always been able to arrange things so that these lectures occur during the time that my students are working on their takehome midterms. This means that they have to come to class sharp and ready to pay attention but that they don't have to do a lot of preparation outside of class. This works well for me.) Students should easily be able to prove Corollary 11.4.10 on their own.[11] I recommend that your students do every problem at the end of this section. If you want to avoid overloading them with out-of-class work, it would work well to have students solve some of the easier problems during class. (For instance, I believe this would work well for problems 1 and 2.) This can be done by breaking students into small groups and getting them to think about the problems for a bit. I predict that some group will fairly rapidly come up with a solution. When all groups have gotten a handle (at least) on what the problems say, some group can be asked to share their solution with the class.

Section 11.5—Existence of the Integral

In this section we get several deep results by exploiting the Cauchy Criteria proved in the previous section. The Riemann integrability of continuous and monotonic functions follow fairly easily. One has to work a bit harder to establish conditions under which the composition of two integrable functions is

[11]However, it might be worth an explicit reminder of what is meant by the word "corollary." I have had some students who think that the thing to do is to try to start from scratch and mimic the proofs of Lemma 11.4.8 and Theorem 11.4.9 to get the upper sum-lower sum criterion!

integrable, but when this is done, the integrability of the product and absolute value of integrable functions follow immediately. The fact that for an integrable function f,

$$\int_a^c f = \int_a^b f + \int_b^c f \text{ (Theorem 11.5.4)}$$

is not all that hard, but there are some technical difficulties. It also follows from the Cauchy Criteria. The proof is outlined in problem 4. The proof that there is an integrable function that is discontinuous at every rational number is outlined in problem 5.

Section 11.6—The Fundamental Theorem of Calculus

The title of the section speaks for itself. At this point, the results are not difficult and should be easily accessible to the students. It is useful to emphasize that the area accumulation function exists and is continuous if f is integrable. But the continuity of f is, in general, required for differentiability of its area accumulation function.

Remark: Excursion I deals explicitly with the question of how integrability of a function is related to regular partitions and regular Riemann sums (e.g. left and right-endpoint Riemann sums over a regular partition). It is very natural to ask why we don't just define our integral in terms of a left or right-Riemann sum. It also easy to answer this question, but the answer is a bit subtle and requires some explanation. I have found this discussion to be enlightening for my students. It can be done by working through technical details, or just by looking at the ideas and discussing the big picture. I recommend it for additional insight into why we do things the way we do.

> Many books begin by defining the integral in terms of upper and lower sums and by-pass Riemann sums altogether. This does simplify the theory, in some ways, but I chose not to do it for two reasons. The main reason is that when students have prior experience with a concept, I want that intuition to be the starting point of the discussion. Because defining upper and lower sums requires least upper bounds and greatest lower bounds, this is not generally the approach taken in a Calculus course and was therefore, from this vantage point, not a good starting point for our discussion. Moreover, the upper sum-lower sum approach moves the discussion completely away from familiar structures, like left and right and midpoint Riemann sums, that are for students the most familiar pieces of the theory.

Chapter 12—Sequences of Functions

Section 12.2—Uniform Convergence

This topic is completely new to most students, so it is a good idea to let them get some intuition for convergence of sequences of functions by having them look at explicit examples of convergent families. Problems 2, 3, 4, 5, 7 and 9 give examples of such families. To get the ball rolling, I usually divide the class into groups and assign a family to each group. Their job is to get a good graph that shows enough terms in the sequence to show the convergence behavior. They then decide (intuitively) what is going on and explain their findings to the class. This is an efficient way for the students in the class to see different sorts of behavior and to begin to make sense of the ideas. I then have everyone write up a rigorous proof of what is going on in one or two of the families.

The main theoretical thrust in this section, however, is to ask whether the limit of a sequence of "nice" functions is also "nice." The examples shown in Section 12.1 show that the pointwise limit of continuous functions is not, in general continuous. The most important theorem in Section 12.2 is, therefore, that uniform convergence *does* preserve continuity (proved in problem 6). Problems 10 and 11 also explore similar themes and are recommended.

Section 12.3—Series of Functions

This is a small excursion into series of functions, a common context for convergence of functions. The Weierstrass M-test is an extremely useful tool for thinking about uniform convergence of series of functions. If you plan to explore series of functions further (e.g. power series in Excursion J or everywhere continuous, nowhere differentiable functions in Excursion K), you should definitely cover this section. If not, the section is optional.

Section 12.4—Interchange of Limit Operations

This section further explores the ways in which uniform convergence is far superior to pointwise convergence from the point of view of being able to draw conclusions about properties of the limit from properties of the sequence. It also puts these ideas in the very important general context of interchanging the order of two limiting processes. There are many times in which we want to exchange the order of limit operations—physicists and engineers do it all the time without a second thought—but mathematics tells us that we have to be careful about this. Hypotheses about the nature of the convergence and of the operations are absolutely necessary. Looking at some examples that illustrate this is very useful for gaining intuition about the difficulties involved. Problem 2 is a good choice. Theorems 12.4.2 and 12.4.4 are the most important theorems in the chapter. Problem 6, which generalizes Theorem 12.4.4, gives a much more useful result and is a good problem to assign, if time permits. (Even power series, which are extremely well behaved, only converge uniformly on compact subsets of their domains.)

Chapter 13—Differentiation: Several Variables

This is the most difficult chapter in the book, both technically and notationally. Thus, in addition to a thorough treatment of the theoretical issues, there are many straightforward problems that help students make connections, problems that ask them to explore enlightening examples or specific cases, and so forth. I apologize for having to say that this chapter has more errors in it than any other. Some have the potential to cause confusion, so please make a special effort to advise your students about the errors before beginning this chapter.

Sections 13.1 and 13.2

These set the stage for the rest of the chapter and can be covered very quickly. Theorem 13.1.2 is a useful theorem and good for getting the students working with vector notation.

Section 13.3—Analysis in Linear Spaces

The chapter assumes familiarity with the basic linear algebraic properties of $\mathbb{R}^n$. Section 13.3 begins with a quick review of those facts. For students that don't have the linear algebra at their fingertips (mine usually don't!) it will probably be sufficient to do a quick, heuristic review of bases, as the only basis that will be considered is the standard unit vector basis. The connections having to do with linear transformations and their properties, including matrix representations, need to be more deeply understood. However, students need not have a thorough grasp of the *proofs* of the facts, instead they need to know and understand the ideas and be able to use them. If you are comfortable with a less rigorous treatment of these ideas, talking about the principles that are discussed on pages 262-266 should be sufficient to get students up to speed so that they can think clearly about analysis in $\mathbb{R}^n$.

The standard norm on $\mathbb{R}^n$ is introduced at the top of page 267 and it is here that analysis really comes into play. Given the students' previous experience with distances and convergence, Exercise 13.3.19 and Theorem 13.3.20 should be straightforward. The end of the section from Lemma 13.3.21 through Lemma 13.3.26 will be tougher going because the idea of the norm of a linear transformation can seem a bit bizarre, at first. The information on norms of linear transformations (Theorem 13.3.20 to the end of Section 13.3) is needed for Section 13.5 on the generalized mean value theorem and for Excursion M on the implicit function theorem. It can be skipped if you don't plan to cover either of these topics.

Section 13.4—Local Linear Approximation for Functions of Several Variables

Differentiability for functions of several variables is finally defined in this section. Corollaries 13.4.9 and 13.4.13, in which connections are made between local linear approximation and rates of change for functions of several variables, are the high points of the chapter. Aside from the proofs of these, problems 1, 3, 4, 5, and 6 establish basic properties of the derivative and are all important. Problems 8 and 10 explore the derivative further in the case of scalar fields. Problem 11 looks at differentiation of parametric curves. Problems 9, 13, and 15 explore differentiation and differentiability in a concrete setting, using specific functions. Problem 16 looks at the relationship between the derivative of a vector-valued function and the derivatives of its scalar components. Problem 18 outlines a proof of the equality of mixed-partials theorem. It is fairly challenging.

Section 13.5—The Mean Value Theorem for Functions of Several Variables

As one moves into deeper territory in the analysis of functions of several variables. These multi-variable generalizations become increasingly important. The results in this section are especially important if you plan to cover the implicit function theorem.

Excursion A—Truth and Provability

This excursion is useful as an accompaniment to the discussion of axioms in Sections 1.2 and 1.3.

Excursion B—Number Properties

This excursion lists a miscellaneous collection of useful numerical results. I chose them more or less based on numerical results that I saw cropping up in the context of other topics. These can be assigned as a collection or individually to supplement the problems given in Chapter 1.

Excursion C—Exponents

This chapter defines and derives exponentiation of real numbers. It is organized as a long set of interconnected exercises. Though one can do this fairly easily by using "high powered" mathematics (e.g. power series, the theory of differential equations), the derivations leave one feeling a bit dissatisfied. It seems as thought one ought not to have to detour so far into (apparently) unrelated theory to get the exponential function. And indeed one does not, as this excursion shows. The approach given here is more elementary and intuitive, if fairly tedious. It is interesting to see, also, that even with this naive approach, the route to irrational exponents requires some surprisingly subtle thinking about uniform continuity and convergence. It is really this last section that is, from the point of view of analysis, the most interesting. And the earlier results, which can take a while, are somewhat "repetitive" in both technique and content. (I had my students work through this excursion once, and Section C.1 took more class time that I thought was warranted for the "payoff.") If you wanted to save some time and get to the more surprising and meaty results in Section C.2, you could assign a representative smattering of earlier portions and then assume the rest of the results

Excursion D—Sequences in $\mathbb{R}$ and $\mathbb{R}^n$

Sections D.1 and D.2—Sequence Convergence in $\mathbb{R}$ and $\mathbb{R}^n$

Proving that actual numerical sequences converge (or not) is, in some sense, peripheral to the broader theoretical concerns of analysis. Nevertheless, it is something we want our theory to be able to support, and it is arguable that it is something we want our students to be able to do. It may seem as though, having proved theoretical results, students will easily be able to undertake the "easier" task of proving that a concrete sequence converges or that it does not. But there are some specialized techniques (tricks?) that come up in this situation, and my experience says that these skills do not come automatically.[12] Theorem D.1.5 is an important theoretical result that relates the convergence of a sequence in $\mathbb{R}^n$ to the convergence of its n-coordinate sequences.

Section D.2 goes into great detail in discussing the relationship between the planning that goes into an "epsilonics" proof and the way that it will eventually be written up as a proof. I usually cover these two sections around the same time that I cover Section 3.4 sequences of real numbers. (It can cover either before or after Section 3.4.)

Section D.3—Infinite limits

This is a tiny section, perhaps not worthy of being set off by itself, but I chose to do that because it *is* dealing with something different. I usually don't cover this explicitly in my classes, but I recommend it as extra reading to my students. Some of them like to think about this because it does speak to something they learned about in their calculus courses.

Section D.4—Some Important Special Sequences

This is more or less a long sequence on inter-connected exercises. The theorems are useful sequence results that show up in important applications such as real number series and (more generally) power series. These can also be assigned as slightly more challenging real number sequence results.

Excursion E—Limits of Functions from $\mathbb{R}$ to $\mathbb{R}$

This continues the task begun in Excursion D. Section E.2 makes some more sophisticated points about the art of "epsilonics." The general principles discussed in that section: "the sum of small things is small," "the product of something bounded and something small is small," and "the multi-task δ" are widely applicable. I often leave my students to read through this on their own and assign problems from the excursion as part of a homework assignment.

[12] This is probably completely obvious to you, but proving that a sequence doesn't converge is usually harder for students than proving that a sequence does converge.

Excursion F—Doubly Indexed Sequences

This is, from my point of view, a *very* optional excursion. However, it does nicely bring together some ideas about uniform convergence if covered after Chapter 12 on sequences of functions. Or, alternatively, it will foreshadow them if it is covered before that chapter. The excursion is really just one long set of interconnected exercises. (Note that the metric space in Theorem F.1.7 must be complete. See the errata.) This set of ideas is useful when considering the multiplication of one series by another.

Excursion G—Subsequences and Convergence

This excursion consists of a sequence of problems about the limit supremum and the limit infimum of a sequence of real numbers. When students first see subsequential limits they seem them as a bit esoteric. But we all know that the limsup and liminf become increasingly important as one progresses into more advanced topics in analysis. Limits infima and suprema are treated here because of their usefulness in the theory of series, which is the subject of Excursion H. A truly serious look at the root and ratio tests requires an understanding of the limsup and the liminf. One would be able to "fudge" and look at less general results for those tests, but I believe that, at this point, students are ready to see beyond the practical tests they saw in calculus to the deeper theoretical under-pinnings that are revealed by the more general theory. (To be precise, it is the characterizations given in problem 4 that are most needed for the proofs of the ratio and root tests.)

Excursion H—Series of Real Numbers

It seems unnecessary to underscore the importance of this excursion. Sections H.4 (rearranging the terms of a series) and H.5 (multiplying series) are a bit esoteric, perhaps,[13] but the ideas treated in Sections H.1, H.2, and H.3 are all really important. This is written in the style of the core chapters: discussion followed by a list of problems at the end of the section. One can pick and choose a bit among the problems but almost all of them either prove an important result or direct the students' attention to something that will help them understand one of the major results better. When I cover this chapter, I end up assigning most of the problems. (Perhaps dividing up the class into groups that will be responsible for presenting specific results.)

Excursion I—Probing the Definition of the Riemann Integral

It is absolutely natural to ask why we don't limit ourselves to looking at "regular" Riemann sums such as right or left endpoint sums on regular partitions. Students can see that in all the examples they look at, they get the right

[13]Though students are fascinated by the result in Section H.4

answer by looking at these "nice" sums. This excursion explores the need for the cumbersome definition that we actually end up using. In the process, it acknowledges the fact that, in practice, we always end up looking at nicely behaved Riemann sums and explains why, despite the more general requirements of the definition, this is actually OK.

Excursion J—Power Series

An extremely important topic. Certainly one I think of as essential in my two-semester course. (I could never get to it in one semester.) The excursion is written like a standard chapter—discussion followed by problems.

Section J.1—Definitions and Convergence

This section establishes the facts about radii of convergence and the nature of that convergence—absolute convergence on the interior of the interval of convergence and uniform convergence on compact subsets of the interval. The exercises help to sharpen the students' understanding of what the theorems say. I think of problems 1 and 3 as essential. Problem 4 says that a power series expansion is unique and is therefore fairly important, as well. Problem 2 is interesting but definitely optional.

Section J.2—Integration and Differentiation of Power Series

By this point, students have seen that term-by-term differentiation and integration of series is not automatic. Thus the nice behavior of power series makes more of an impression on real analysis students than it does on calculus students. The only problem that I don't think of as absolutely essential is problem 5, though even it is a very useful theorem. Lemma J.2.1 (proved in problem 1) requires knowledge of a couple of real number sequential limits. The book gives dispensation for just assuming these facts. If you want to keep a tighter chain of reasoning, the proofs will require a short "detour." The proofs are not terribly difficult (hints are given in Excursion D.4.5), but neither are they completely trivial.

Section J.3—Taylor Series

This section rounds out the information on power series by talking about the relationship between a function with derivatives of all orders at $x = a$ and its Taylor series based at a. There is no "conclusive" theorem proved. However, with Corollary J.3.3, students can show that the sine, cosine, and exponential functions are all analytic (problems 1 and 2). Problem 3 leads students through a proof that shows that the existence of derivatives of all orders at a point does not guarantee the analyticity of the function.

Excursion K—Everywhere Continuous, Nowhere Differentiable

This begins with a prefatory discussion and ends with a detailed outline of the proof that a certain function is everywhere continuous and nowhere differentiable. I have successfully used this excursion as a project in which students worked independently to tease out the details of the proof and then wrote up a nice, self-contained paper discussing the idea of an everywhere continuous, nowhere differentiable function and giving a proof of the existence of such a function.

Excursion L—Newton's Method

In Section L.2 we get a lot of mileage out of the results in Chapter 10, and we do so without much increase in the difficulty. So this is something that I often do soon after covering Chapter 10. Sections L.3 and L.4 show that difficulties can arise with Newton's method and that care must be taken in the original choice of x_0. Section L.5 states and proves the standard quadratic error estimate for Newton's method.

Excursion M—The Implicit Function Theorem

This is the hardest chapter in the book. The implicit function theorem is a truly deep theorem of mathematics. Students face three big obstacles. The first is simply to understand the statement of the theorem. There are so many hypotheses that it is hard to take them all in. It is even harder to understand why they all have to be there. Next, they have to understand the details of a difficult proof. (There are various ways of proving this theorem. They are all hard.) Then they have to understand the motivation/intuition behind the proof. It's all difficult.

But there are some truly beautiful ideas in this chapter. The proof uses iteration in a clever way. The connections to Newton's method are lovely, and (when all is said and done) give us a nice intuition for the proof. But it *is* hard.

The only time I got far enough with my students to cover this chapter, we were running short on time, so I just lectured on the chapter. I would prefer to have a combination of lecture and group work, in which we intersperse general discussion of the ideas with small groups working on specific aspects and details in the proof of the theorem. (The same applies to further theorems like the continuity and differentiability of the solution theorem and the inverse function theorem.)

Excursion N—Spaces of Continuous Functions

This excursion opens some vistas into deeper study of analysis. Students get a glimpse of the power of abstraction when they make the connection between compactness in $C(K)$ and the question: "when can we extract a uniformly convergent subsequence from a sequence of continuous, real-valued functions on a compact metric space?" The Stone-Weierstrass theorem connects the idea of a dense subset with polynomial approximations of an arbitrary continuous function. The fact that the same theorem gives us a parallel result for approximation by sines and cosines, brings the whole lesson to a sharp point.

Excursion O—Solutions to Differential Equations

This excursion makes use of iteration in $C(K)$ spaces to deduce the existence and uniqueness of solutions to a certain class of differential equations, bringing many of the themes of the book together. It is less difficult than the excursion on the implicit function theorem, but is just as deep and beautiful an application of iterative methods.

Part IV

Errata

This document contains the errors *Closer and Closer: Introducing Real Analysis*, as I know them at this time. There is a list of substantive errors at the beginning of the document and a list of typographical errors at the end. There will always be an up-to-date list of errors available on my website. Feel free to e-mail me for a link if that is easiest for you. Moreover, if you find errors that are not on this list, I would be very grateful if you were to let me know about them. (Schumacherc@kenyon.edu.)

0.2 Functions

- Page 24—Problems 2(c) and 2(d) should read thus (I have underlined the words that need to be changed. The underline should not appear in the text.)

  (c) If $g \circ f$ is onto, then f is <u>onto</u>.

  (d) If $g \circ f$ is onto, then g is <u>onto</u>.

1.4 Least Upper Bound Axiom

- Page 60—Problem 3 is false, as stated. t must be non-negative.

  Original phrasing:

  Let $t \in \mathbb{R}$ and let $S \subset \mathbb{R}$ that is bounded above.

  Suggested rephrasing:

  Let $t \in \mathbb{R}^+$ and let $S \subset \mathbb{R}$ that is bounded above.

- Page 60—In problem 4, the sets S and T need to be non-empty. The problem should say, "Let S and T be non-empty subsets of $\mathbb{R}$ that are bounded above.

2.2 The Euclidean Metric on $\mathbb{R}^n$

- Page 65—Middle of the page. The definition of the metric on $\mathbb{R}^n$ is mistypeset as a fraction. It should read:

  $$d((a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n)) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + \ldots + (a_n - b_n)^2}.$$

3.4 Sequences in $\mathbb{R}$

- Page 92— In problem 10, $\lim_{n\to} c^{\frac{1}{n}} = 1$ should be $\lim_{n\to\infty} c^{\frac{1}{n}} = 1$.
- Page 93—In problem 11, the reference to Excursion D.4.7 should really be a reference to Exercise D.4.7.

3.7 Open Sets, Closed Sets, and the Closure of a Set

- Page 99—In part 5 of Exercise 3.7.4 the statement "Let $x \in \overline{X}$" should, instead, be "Let $x \in X$."

4.3 Continuous Functions

- Page 114—The point a referred to in problem 7 must be a limit point of X, otherwise the limit is not defined.

  Original phrasing:

  ... Prove that f is continuous at $a \in X$ if and only if ...

  Should be:

  ... Prove that f is continuous at a limit point a of X if and only if ...

4.4 Uniform Continuity

- Page 116—For problem 6(c). Add the parenthetical statement

  > (Assume for now that the difference of two continuous, real-valued functions is continuous. This will be proved in Section 5.3.)

  at the end of the statement of the problem.

5.1 Limits, Continuity, and Order

- Page 122—Second paragraph after **Some Useful Special Cases**. The sentence with the bad reference (??) should be: "Corollary 5.1.9 is a special case of Theorem 5.1.1."

5.3 Limits, Continuity, and Arithmetic

- Page 127—In theorem 5.3.1(4), the statement reads "Assume $g(x) \neq 0$ on some interval containing $a \ldots$" it should, instead, be "Assume $g(x) \neq 0$ on some open set containing $a \ldots$"

7.1 Compact Sets

- Page 143—In part (b) of problem 17. The set X should be

$$X = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1, x > 0 \text{ and } y > 0\}.$$

8.1 Connected Sets

- Page 153—Problem 1. The problem reads,

  > Prove the IVT for a continuous function $f : [a, b] \to \mathbb{R}$ as follows. Suppose that γ is between $f(a)$ and $f(b)$. Let $c = \sup\{x \in [a, b] : f(x) \leq \gamma\}$. Show that $f(c) = \gamma$.

  It should, instead, read

  > Prove the IVT for a continuous function $f : [a, b] \to \mathbb{R}$ as follows. Suppose that $f(a) \leq \gamma \leq f(b)$. Let $c = \sup\{x \in [a, b] : f(x) \leq \gamma\}$. Show that $f(c) = \gamma$. Modify the argument for the case when $f(b) \leq \gamma \leq f(a)$.

9.2 The Derivative

- Page 163— In the first line of the second paragraph contained in the box, "$y \to 0$" should be "$y \to x$."

- Page 163—In the last line in the box "the expression" should be "the expression in Theorem 9.2.2."

9.7 Polynomial Approximation and Taylor's Theorem

- Page 184—In the proof of Theorem 9.7.1, third line from the end. The line reads: "But $A''(x) = f'' - M$, so $A''(c) = 0$ which implies that $M = f''(c)$"

  It should, instead, read "But $A''(x) = f'' - M$. So $A''(c) = 0$ implies that $M = f''(c)$"

10.1 Iteration and Fixed Points

- Page 197—Problem 8, second line reads " $\ldots 1 \leq k \leq 3 \ldots$." It should, instead, read " $\ldots 1 < k < 3$."

11.4 Families of Riemann Sums

- Page 223—Last line of the page reads:

  $$N^*(z_j - z_{i-1}) \text{ where } N^* = \sup\{f(x) : x \in [z_i, z_j]\}.$$

  It should read, instead,

  $$N^*(z_j - z_i) \text{ where } N^* = \sup\{f(x) : x \in [z_i, z_j]\}.$$

11.5 Existence of the Integral

- Page 228—The definition of the function in Example 11.5.3 needs to be modified slightly. Add "or 0" in the first line of the definition:

  $$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational or } 0 \\ \frac{1}{q} & \text{if } x \text{ is rational and } x = \frac{p}{q} \end{cases}$$

- Page 228—Theorem 11.5.4. The third sentence reads "If $f : K \to \mathbb{R}$ is a function, then the integral $\int_a^b f \ldots$." It should, instead, read "If $f : K \to \mathbb{R}$ is a function and $a < c < b$, then the integral $\int_a^b f \ \ldots$"

- Page 232—The expression at the bottom of the page reads.

  $$\left| \mathcal{R}(f, P) - \left(\int_a^c + \int_c^b f \right) \right|.$$

  It should read, instead,

  $$\left| \mathcal{R}(f, P) - \left(\int_a^c f + \int_c^b f \right) \right|.$$

- Page 233—Problem 4(c) currently reads: "Now use the result from parts (a) and (b) to remove the restriction that $a < b < c$. (You may need to break this into several cases.)" It should, instead say "Assume any two of the three integrals $\int_a^c f$, $\int_c^b f$ and $\int_a^b f$ exist. Use the result from parts (a) and (b) to remove the restriction that $a < c < b$. (You may need to break this into several cases.)"

- Page 233—The definition of the function in Problem 5 needs to be modified slightly. Add "or 0" in the first line of the definition:

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational or } 0 \\ \frac{1}{q} & \text{if } x \text{ is rational and } x = \frac{p}{q} \end{cases}$$

  And the word "non-zero" needs to be added to the statement in part (a): "Show that f is discontinuous at every non-zero rational number."

12.2 Uniform Convergence

- Page 245—In problem 12, the next to the last line before part (a). " …that $\lim_{m\to\infty} = f(x)$" should, instead, read "…that $\lim_{m\to\infty} f(x_m) = f(x)$."

13.1 What Are We Studying

- Page 259—Theorem 13.1.2(1). The second line reads "… $\lim\limits_{k\to\infty} \mathbf{f}(\mathbf{y}_n) = \mathbf{b}$ if and only if for each $i = 1, 2, \ldots, m$, $\lim\limits_{k\to\infty} f_i(\mathbf{y}_n) = b_i$." the subscripts n should, instead be k's. The line should read "… $\lim\limits_{k\to\infty} \mathbf{f}(\mathbf{y}_k) = \mathbf{b}$ if and only if for each $i = 1, 2, \ldots, m$, $\lim\limits_{k\to\infty} f_i(\mathbf{y}_k) = b_i$."

13.3 Analysis in Linear Spaces

- Page 263—In Exercise 13.3.3, the upper limit on the displayed equation should be k. It should read

$$\mathbf{v} = \sum_{i=1}^{k} a_i \mathbf{b}_i \text{ instead of } \mathbf{v} = \sum_{i=1}^{n} a_i \mathbf{b}_i.$$

- Page 265—Theorem 13.3.9(3). The linear transformation referred to in this part of the problem needs to be one-to-one. The problem should read: "Suppose that $\mathbf{L}$ is one-to-one, then the set $\{\mathbf{L}(\mathbf{e}_i)\}$, the image of the standard basis, is linearly independent in $\mathbb{R}^m$."

- Page 267-268—In Theorem 13.3.20. Because n is the dimension of the space in this problem, every subscript n should be changed to an i. The theorem should read:

  Let $(\mathbf{x}_i)$ and $(\mathbf{y}_i)$ be sequences in $\mathbb{R}^n$ converging to $\mathbf{x}$ and $\mathbf{y}$, respectively. Let (t_i) be a sequence in $\mathbb{R}$ that converges to a scalar t. Let k be an arbitrary scalar. Prove the following facts:

  1. $(k\mathbf{x}_i)$ converges to $k\mathbf{x}$.
  2. $(t_i\mathbf{x}_i)$ converges to $t\mathbf{x}$.
  3. $(\mathbf{x}_i + \mathbf{y}_i)$ converges to $\mathbf{x} + \mathbf{y}$.
  4. $(\mathbf{x}_i \cdot \mathbf{y}_i)$ converges to $\mathbf{x} \cdot \mathbf{y}$.

- Page 270—In Problem 10—Because n is the dimension of the space in this problem, every subscript n should be changed to an i. The problem should read

  Let $(\mathbf{x}_I)$ be a sequence in $\mathbb{R}^n$ and let (t_i) be a sequence of scalars.

  (a) Suppose that $(\mathbf{x}_i)$ converges to $\mathbf{0}$ and that (t_i) is bounded in $\mathbb{R}$. Prove that $(t_i\mathbf{x}_i)$ converges to $\mathbf{0}$.

  (b) Suppose that (t_i) is a sequence in $\mathbb{R}$ that converges to 0, and $(\mathbf{x}_i)$ is a bounded sequence in $\mathbb{R}^n$. Prove that $(t_i\mathbf{x}_i)$ converges to $\mathbf{0}$.

- Page 271—Problem 12, first line—"established in Lemma 13.3.21 ..." should, instead, read, "... established in Corollary 13.3.23."

- Page 271—In problem 15(a). There is a typographical error in the description of $\mathrm{B}_r(\mathbf{x})$. It reads "$\mathbf{x} + s\mathbf{u} : 0 \leq s \leq r \ldots$." It should, instead, read "$\mathbf{x} + s\mathbf{u} : 0 \leq s < r \ldots$."

- Page 271—In problem 16, second line: it reads " ... if and only if $\mathbf{L}(\mathbf{e}_i) = \mathbf{S}(\mathbf{e}_i) \ldots$." It should, instead, read "... if and only if $\mathbf{T}(\mathbf{e}_i) = \mathbf{S}(\mathbf{e}_i) \ldots$."

13.4 Local Linear Approximation

- Page 277—In Theorem 13.4.8—last line before the equation at the end reads "it follows that for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, ..." The last quantification is unnecessary. It should read "it follows that for $i = 1, 2, \ldots, n$, ..."

- Page 279—In Theorem 13.4.12, in "Step 1." The very beginning reads "Define $r : E \to \mathbb{R} \ldots$." It should, instead, say "Define $e : E \to \mathbb{R} \ldots$."

- Page 283—Problem 8. The displayed equation at the very end should read,

$$f(\mathbf{x}) = \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + f(\mathbf{a}) + r(\mathbf{x}), \qquad \text{and } \lim_{\mathbf{x} \to \mathbf{a}} \frac{|r(\mathbf{x})|}{\|\mathbf{x} - \mathbf{a}\|} = 0.$$

- Page 284—Problem 13. The function is incorrect. It should, instead, be

$$f(x,y) = \begin{cases} \dfrac{yx + 2xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases} .$$

- Page 286—The displayed expression near the bottom of the page is off by a minus sign.

$$D_{\mathbf{1}} D_{\mathbf{2}} f(\mathbf{a}) \approx \frac{\frac{f(a_1,a_2+k)-f(a_1,a_2)}{k} - \frac{f(a_1+h,a_2+k)-f(a_1+h,a_2)}{k}}{h}$$

  should, instead, be

$$D_{\mathbf{1}} D_{\mathbf{2}} f(\mathbf{a}) \approx \frac{\frac{f(a_1+h,a_2+k)-f(a_1+h,a_2)}{k} - \frac{f(a_1,a_2+k)-f(a_1,a_2)}{k}}{h}.$$

Excursion D.4 Some Important Special Sequences

- Page 317—Middle of the page (Step 2. in the proof sketch for Theorem D.4.5). "Excursion 3" should instead be "Excursion C."

Excursion F.1 Double Sequences and Convergence

- Page 328—In definition F.1.6, 4th line reads, "... $N \in \mathbb{N}$ such that for all $m > N$ and all $N \in \mathbb{N}$ ...." It should, instead, read " ... $N \in \mathbb{N}$ such that for all $m > N$ and all $n \in \mathbb{N}$ ..."
- Page 329—The metric space in Theorem F.1.7 needs to be complete. In other words, the hypothesis should read "Let X be a complete metric space, ..."

Excursion H.1 Series of Real Numbers

- Page 336—Theorem H.1.5 in the last line before the displayed inequality, "$n > m > N$" should, instead, read "$n \geq m > N$."

Excursion H.4 Rearranging the Terms of a Series

- Page 353—As stated in Lemma H.4.5, the last word in problem 2 should be "diverge" not "converge."

Excursion I.1 Regular Riemann Sums

- Page 359—The second displayed expression reads

$$\sum_{i=0}^{n-1} f(x_i)(x_i - x_{i-1}).$$

It should, instead, be

$$\sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1})$$

Excursion J.1 Power Series

- Page 366—The second and fourth power series given in Exercise J.1.6 should start at $n = 1$:

$$\sum_{n=0}^{\infty} \frac{3}{n^2}(x-5)^n \text{ should instead be } \sum_{n=1}^{\infty} \frac{3}{n^2}(x-5)^n.$$

$$\sum_{n=0}^{\infty} \frac{1}{n}(x-5)^n \text{ should instead be } \sum_{n=1}^{\infty} \frac{1}{n}(x-5)^n.$$

Excursion M.4 The Inverse Function Theorem

- Page 407—in problem 2, the second line, "The reverse is also possible" should instead read "The reverse is also possible provided that we assume all of the partial derivatives of **F** exist and are continuous." The next to the last line of the problem should read "... indeed, equivalent under the hypothesis that all partial derivatives exist and are continuous, assume ..."

Excursion N.3 The Stone-Weierstrass Theorem

- Page 419—in step 2 of problem 4(b). The text should read: "Let $x \in [0, 1]$. Notice that if $f(x) = x$, then $f(x) = f(x) - (f(x))^2 + x^2$. In fact, $f(x) = x$ is the unique non-negative fixed point for the function $F : C[0, 1] \to C[0, 1]$ given by $F(f) = f - f^2 + q$ where q is the quadratic function $q(x) = x^2$ on $[0, 1]$."
- Page 420—Steps 3 and 4 should be reversed.

Excursion O.2 Picard Iteration

- Page 426—In problem 1, U must be convex. The problem should read, "Let $U \subseteq \mathbb{R}^2$ be convex, and let $f : U \to \mathbb{R} \ldots$"
- Page 426—In problem 2. The displayed equation should match the corresponding equation on the previous page:

$$F(y)(t) = x_0 + \int_{t_0}^{t} f(u, y(u))\, du.$$

Excursion O.3 Systems of Equations

- Page 430—In problem 5. The constant α mentioned in the result should be $\alpha = \min\left\{r, \frac{m}{nM}\right\}$.

Less important errors (more in the way of typos.)

- Page 80—In problem 11, the $\overline{X}$ should just be X.
- Page 108—The word "appoaches" in Theorem 4.2.3 should, instead, be "approaches."
- Page 112—There should be a period at the end of the displayed equation on the very last line.
- Page 132—The square brackets around the Hint at the end of problem 3(b) should, instead, be parentheses.
- Page 142—In problem 11, third line: "susequences" should be "subsequences."
- Page 160—In the last paragraph, end of the first line, there is a comma after the word countably many. This comma shouldn't be there.
- Page 210—In the footnote at the bottom of the page, second line. There should be a close parenthesis after the reference [McL].
- Page 212—In Theorem 11.2.3, in the second line we see "...that is a Riemann integrable on ...". It should read, instead,"...that is Riemann integrable on ...."
- Page 212—In Theorem 11.2.3, in the second line we see "...that is a Riemann integrable on ...". It should read, instead,"...that is Riemann integrable on ...."
- Page 244—The last line of the page needs a space between "functions" and "on."

- Page 245—In problem 12, third line. The sentence at the end of the line that begins "The for all . . . " should instead begin with "Then for all . . . ."
- Page 252—The first line of Corollary 12.4.5 ends in the word "is" and should, instead, end in the word "are."
- Page 277—Theorem 13.4.8: The function referred to is a vector-valued function. Thus in lines 2 and 3, f should instead be $\mathbf{f}$.
- Page 282—Problem 3. The function f mentioned in the first line should, instead be $\mathbf{f}$, as it is a vector-valued function.
- Page 330—In problem 1, there should be a comma between "*non*-convergent" and "bounded."
- Page 305—In Theorem C.1.2, the first line should read "Let a be a positive real number. Let r and s . . . "
- Page 361—In problem 2, "Reimann" should, instead, be "Riemann."