Articulation Issues: High School to College Mathematics

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The Mathematical Education of Teachers II (MET II) (CBMS, 2012) chapter on high school teachers opens by noting that the double discontinuity experienced by prospective high school mathematics teachers and described by Felix Klein (1908) still exists today. As stated in MET II, this double discontinuity consists of the jolt experienced by the high school student moving from high school to university mathematics, followed by a second jolt moving from the mathematics major to teaching high school (p. 53). MET II addresses ways of smoothing the second jolt, but both jolts and extensions will be considered here, as they are the essence of the articulation issues between school and college mathematics.

This article is being written as the Common Core State Standards in Mathematics (CCSSM, 2010) are being implemented in almost all states in the U.S. Consequently, there is little evidence beyond speculation as to how CCSSM will impact the transition from school to college mathematics. The impact of the new standards and the associated assessments will not be evident for several years, but inertia in the K-20 education system will likely prevent major changes.

Readiness for College Mathematics

As in most state standards that preceded CCSSM, and in college admissions testing by ACT and the College Board, readiness for college mathematics has been a much discussed and sought after goal. For example, ACT has a benchmark score of 22 on the mathematics test for readiness for college mathematics (approximately 560 on SAT mathematics reasoning). This definition of college readiness is narrow, focusing only on the likelihood of making a grade of C or better in the first degree-credit bearing college mathematics course, often college algebra. CCSSM aims at college and career readiness, built on the principle that all students will meet the standards laid out for high school mathematics as with the NCTM 2000 standards. The assessments associated with CCSSM are being developed by two consortia of states to be implemented in the 2014-2015 school year. The CCSSM comprehensive assessment results, scheduled for the last part of grade 11, should therefore be available for consideration for the entering class of college students for 2016-2017. Some higher education institutions and systems in the states that signed on to CCSSM have agreed to use the CCSSM assessment results as a measure of readiness for college level mathematics, analogous to many institutions currently using the results of the ACT mathematics score or the SAT mathematics reasoning score. The agreements to use the CCSSM assessment scores in this way typically have been made at upper administrative college and university levels and not by mathematics departments. How mathematics faculties will accept and react to CCSSM assessments is largely unknown. In any event, validation of these assessment results as a reliable measure of readiness for college mathematics is likely half a dozen years or more away at this point.

However, the practice of placing entering college students in one of several possible entry-level mathematics courses is not likely to change with the implementation of CCSSM (as outlined above). Whether a student is prepared for calculus, for precalculus, for college algebra, or some other entry-level course likely will not be determined by CCSSM assessments. Therefore, placement programs will still be necessary.

What do we know? We know that for the near future, placement programs will continue to be needed and will probably not change.

What would we like to know? We would like to know how the implementation of CCSSM will affect the mathematical preparation of entering college students and what the CCSSM comprehensive assessments will tell us about these students. In addition, what changes might be needed in current placement tools to better align outcomes with the mathematics students will be expected to learn in high school.

What resources are available? College Board and ACT each offer placement testing systems, Accuplacer and COMPASS, respectively. The MAA placement tests are offered by Maplesoft and consist of the traditional tests in basic algebra, advanced algebra, elementary functions, trigonometry, and calculus readiness. The newest (2010) test is the Calculus Concept Readiness (CCR) instrument, and another new test, Algebra and Precalculus Concept Readiness (APCR), is being field tested in 2013 -2014. Other placement testing systems are available from publishing companies, and some, e.g. ALEKS, have tutorial systems included.

Including Remediation[1] in College Algebra

One of the continuing articulation issues is that of requiring developmental courses in colleges and universities. For example, in Arkansas, state regulations require that students who have an ACT mathematics score less than 19 must complete a developmental mathematics course prior to enrolling in a degree-credit bearing mathematics course. This results in more than 40% of the entering college students being placed in developmental mathematics courses, namely courses that are prerequisites for college algebra. Since ACT sets as a benchmark an ACT mathematics score of 22, that creates a band of scores 19-22 that ACT does not believe indicate college readiness, yet Arkansas policies say otherwise. In response to this, some institutions (e.g. University of Arkansas, Fayetteville) have created an alternate college algebra course with more class time and more support for students with ACT scores of 19-22. This has proved far more efficient and effective than placing these students in a developmental course and then expecting them to finish college algebra.

[1] We are adhering to the distinctions between remedial and developmental as outlined, for example, by Ross (1970). Remedial instruction provides instruction in prerequisite material that is not a part of the course's objectives. Developmental courses have specific learning objectives that are required of subsequent courses, e.g. college algebra.

Some placement examination systems (e.g. ALEKS) have instituted ways to provide learning tutorials to move the student from one placement level to a higher level. Often the purpose is analogous to the college algebra with support scheme above, namely to move the student from a developmental placement to one that is degree credit bearing.

What do we know? We know that developmental mathematics courses in college are minimally successful in moving traditional age college freshmen from being unsuccessful in mathematics to being successful.

What would we like to know? We would like to know better ways to improve learning of mathematics in high school. Better yet, we would like to know how to better motivate students to learn mathematics, especially algebra, the first time they study it.

Calculus and Precalculus

Attrition from the algebra-precalculus-calculus sequence is known to be significant and affects the number of students in the science, technology, engineering and mathematics (STEM) pipeline. As these courses overlay the intersection of high school and college mathematics, clarity and consistency in content and cognitive demands are needed for good articulation and realization of the potential for understanding in the next course and beyond. Although the content of precalculus courses may be the same, the cognitive demands of courses can be very different. Recently, the content and cognitive demands of algebra, precalculus, and calculus courses have been studied. As Carlson, Oehrtman, and Engelke (2010) point out, "there is now substantial research on what is involved in learning key ideas of algebra through beginning calculus. However, a cursory examination of the commonly used curricula suggests that this research knowledge has had little influence on precalculus level curricula" (p. 114). (See also, Tallman & Carlson, 2012.)

Currently, an NSF-funded project of the Mathematical Association of America (MAA) is aimed at using the research results pointing to conceptual understanding needed to succeed in algebra, precalculus, and calculus to construct placement tests to measure this essential understanding. The first of these tests, the Calculus Concepts Readiness (CCR), is being used by some institutions to test for calculus readiness. Preliminary results indicate that many beginning calculus students do not have strong understandings of fundamental concepts, the major one being that of a function, especially viewed as a process. Of the twenty-five multiple choice precalculus items on CCR, students in calculus 1 at major universities on average get fewer than half of them correct. Moreover, the results are similar when CCR was administered to a sample of a couple hundred high school mathematics teachers, mostly teachers of algebra and precalculus. A second test, Algebra and Precalculus Concept Readiness (APCR), based on the same research, is nearing completion.

The fact that many students in the calculus 1 courses in college are achieving passing grades without having strong understandings of seeming essential precalculus concepts suggests that the research results may be wrong. However, it more likely points to the lack of cognitive demands of the calculus courses themselves. This latter point is supported by the results of Tallman and Carlson (2012) from examining a sample of 150 final examinations from college and university calculus 1 courses. They determined that about 15% (slightly different for different kinds of institutions) of the examination items required understanding of concepts or applying understanding of concepts. That means that 85% of the items required recall of information or recall and application of a procedure. Interestingly, a similar examination of Advanced Placement (AP) calculus free-response items found that approximately 40% of the items required applying conceptual understanding. The fact that performance on the AP examination has a different scale for determining grades dilutes this comparison and does not necessarily indicate that the AP calculus students exhibit stronger conceptual understanding of calculus than the university calculus students.

The above points strongly to weaknesses in the algebra-precalculus-calculus sequence caused by lack of cognitive demand. Surely, these weaknesses cause difficulties in subsequent STEM courses, indicating a lack of articulation between school and college or within colleges themselves. Another place that this lack of articulation surfaces is within the mathematics major. Many mathematics majors experience a jolt when they move from the more methodological calculus courses into the more abstract courses in algebra and advanced calculus. In fact, many (if not most) mathematics departments have instituted bridge courses (e.g. introduction to proof) to soften this jolt. Computer-based homework systems that provide testing using multiple choice items, often used in courses up through the calculus sequence, can aggravate this jolt, as the cognitive demand of such computer managed courses is often well below that of a junior-level course in abstract algebra or advanced calculus.

What do we know? Strong evidence suggests that the algebra-precalculus-calculus sequence, whether in high school or college, is not meeting its potential for use in subsequent courses or in preparing students, particularly mathematics majors, for smooth transitions to more abstract and advanced study.

What would we like to know? We would like to know how to influence schools and colleges to offer precalculus and calculus courses that are more cognitively demanding.

Differing Systems and Pedagogies

High school mathematics classrooms often differ from college and university classrooms. Most high school mathematics classes are in interactive classrooms with less than 30 students. Many incorporate collaborative learning situations, frequently with inquiry-based instruction. Contrast that with a lecture style university classroom with more than 100 students, sometimes many more than 100. This system of large lecture-style classes, present in many large universities, is not only different from the system in most high schools, but it is also inconsistent with what research in learning theory tells us that is most effective for long-term retention and transfer, which provides another instance where research results are not significantly changing classroom practices.

These differences will potentially increase with the use of online courses and degree programs in colleges and universities. The potential of delivering high-quality instruction by expert teachers to all corners of the world is indeed attractive, but many questions remain about promoting interaction and keeping the cognitive levels of grading high. Some of these questions are raised in the following section on what is known about how people learn best.

Ignoring How People Learn Best for Long-Term Retention and Transfer
The expanded edition of *How People Learn* (Bransford, Cocking, & Brown, 2001) reported research results on learning and how these results can improve teaching and learning. Subsequent to the publication of *How People Learn*, Diane Halpern and Milton Hakel (2003) reported the results of a consensus agreement among 30 experts on the science of cognition in "Applying the Science of Learning to the University and Beyond." They summarized the findings by giving ten basic laboratory-tested principles (listed in brief below) needed for enhancing long-term retention and transfer. In the opening paragraphs Halpern and Hakel (2003) write, "We have found precious little evidence that content experts in the learning sciences actually apply the principles they teach in their own classrooms. Like virtually all college faculty, they teach the way they were taught. But, ironically (and embarrassingly), it would be difficult to design an educational model that is more at odds with the findings of current research about human cognition than the one being used today at most colleges and universities" (pp. 37-38). So, many of us in collegiate mathematics are unaware of or ignoring the research results on what concepts students need to understand to be successful in calculus, and we are seemingly joined in this by our high school colleagues. However, high school classroom practices are much more in tune with the ten Halpern and Hakel (2003) principles than are most college classrooms.

1. The single most important variable in promoting long-term retention and transfer is "practice at retrieval."
2. Varying the conditions under which learning takes place makes learning harder for learners but results in better learning
3. Learning is generally enhanced when learners are required to take information that is presented in one format and "re-represent" it in an alternate format.
4. What and how much is learned in any situation depends heavily on prior knowledge and experience.
5. Learning is influenced by both our students' and our own epistemologies.
6. Experience alone is a poor teacher. Too few examples can situate learning. Many learners don't know the quality of their comprehension and need systematic and corrective feedback.
7. Lectures work well for learning assessed with recognition tests, but work badly for understanding.
8. The act of remembering itself influences what learners will and will not remember in the future. Asking learners to recall particular pieces of information (as on a test) that have been taught often leads to "selective forgetting" of related information that they were not asked to recall.
9. Less is more, especially when we think about long-term retention and transfer. Restricted content is better.
10. What learners do determines what and how much is learned, how well it will be remembered, and the conditions under which it will be recalled.

What do we know? We know what research on learning tells us about teaching for long-term retention and transfer. We also know that most college mathematics faculty members do not apply the principles gleaned from this research in their classrooms.

What do we need to know? We need to know how to teach – both in high school and college – to maximize long-term retention and transfer. There are research opportunities here.

Statistics Articulation

The situation in statistics articulation between high school and college is far less structured than that in mathematics. This is due in large part to the relative newness of statistics to the K-12 curriculum and its growing presence in undergraduate studies. However, with the growth of AP statistics over the past 15 years and the more definitive inclusion of probability and statistics in CCSSM, articulation possibilities are increasing. Currently, many college and university statistics courses cover content that is included in CCSSM, and developmental courses in statistics are rare. So are placement examinations in statistics. However, with a more determined effort via CCSSM to include and assess statistics in grades 6-12, entry-level college statistics courses have a chance to build on previous knowledge. In colleges and universities, statistics teaching is frequently dispersed across several departments – social sciences, agriculture, engineering, business, and mathematics. This most likely means that the effects of increased attention to statistics in K-12 because of CCSSM will be delayed a bit longer as these non-mathematics disciplines are likely to be be slower to recognize the changes.

Teachers from High School to College

This is the jolt that is directly addressed by MET II. As noted by Klein (1908), this jolt stems mostly from the lack of any apparent connections between the mathematics the teachers studied as mathematics majors and the mathematics and statistics that they are expected to teach. There is no good reason for this lack of connections. Geometry, history of mathematics, abstract algebra, probability and statistics, and functions and relations form the foundations of school mathematics. Most importantly they give teachers the conceptual frameworks on which to hang their facts. Providing these conceptual frameworks, or versions thereof, to their students can bring coherence to learning, thereby promoting long-term retention and transfer. As concluded in *How People Learn*, "To develop competence in an area of inquiry, students must: (a) have a deep foundation of factual knowledge, (b) understand facts and ideas in the context of a conceptual framework, and (c) organize knowledge in ways that facilitate retrieval and application."

Prospective teachers would be better able to teach if their undergraduate mathematics courses modeled what is known about effective teaching for long-term retention and transfer. Some of the more prominent works in this area include the eight CCSSM standards for mathematical practice, the ten principles from Halpern and Hakel (2003) given above, the five elements[2] of effective thinking by Burger and Starbird (2012), the six core competencies for quantitative reasoning (Boersma, et al.,2011), and the conclusions reached in *How People Learn*. Although the eight CCSSM standards for mathematical practice were derived from the five strands of mathematical proficiency from *Adding It Up* (National Research Council, 2001) and the five NCTM (2000) process standards, these alternate expressions help in understanding the practices in slightly different ways. For example, the *Adding It Up* strand of productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence in one's own efficacy) is very important to keep in mind whether teaching college students or K-12 students.

What do we know? We know that most college and university teachers teach the way they were taught and in ways that are different from the ways that high school teachers will be expected to teach.

What would we like to know? We need more examples of ways to effectively connect the undergraduate mathematics major courses to school mathematics.

What resources are available? MET II is available online and outlines the CBMS recommendations for the education of teachers. The references given here – *How People Learn*, *Adding It Up*, *Five Elements of Effective Thinking* (Burger and Starbird), Halpern and Hackel's *Change* paper, CCSSM's eight standards for mathematical practice, and the six core competencies for quantitative reasoning (Boersma, et al.) are excellent readings for helping prospective teachers develop.
Two books that connect undergraduate mathematics to school mathematics are *Mathematics for High School Teacher: An Advanced Perspective* by Usiskin, Peressini, Marchisotto and Stanley and *An Introduction to Abstract Algebra with Notes to the Future Teacher* by Nicodemi, Sutherland, and Towsley.

Quantitative Reasoning Courses
Quantitative reasoning (QR) or quantitative literacy (QL) courses are increasing as the newest entry in the mathematics courses for general education. Since these courses rely most heavily on proportional reasoning and number sense that is developed mostly in middle school mathematics and not emphasized in high school, there can be an articulation issue when college students encounter these. One of the weaknesses of CCSSM is its attention to the third readiness "C," citizenship readiness, to go along with college and career. High school courses have not emphasized general education issues, being hard pressed to cover all the content in geometry, algebra, statistics, and functions. However, teachers are beginning to use applications to the everyday worlds of their students as motivation for the students to learn mathematics. This has become more possible because of the increased number of quantitative issues in contemporary society and is especially effective in teaching statistics and data analyses.

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[2] See appendix for the lists of the elements of effective thinking, the strands of mathematical proficiency, the core competencies for quantitative reasoning, the eight CCSSM practice standards, and the conclusions from *How People Learn*.

What do we know? We know that courses in QR and QL are increasingly offered at many colleges and universities, sometimes by units other than mathematics departments.

What would we like to know? We would like to know how to evaluate QR and QL courses that are unusual in mathematics departments because they are not defined by their content as are most mathematics courses.

What resources are available? The National Numeracy Network (NNN) (an interdisciplinary organization) and the QL SIGMAA of MAA can provide information about QR and QL courses. NNN publishes a free access online journal, *Numeracy*, twice annually containing papers on QR and QL education.

College Credit for Courses Taken while in High School

There are two types of courses taken by high school students that may earn them college credit. The first are courses that are validated by examinations: Advanced Placement (AP) courses by the College Board and International Baccalaureate (IB) courses. Students entering a college or university and wanting credit for either AP or IB courses should have their grades in those courses reported to the college or university. Individual colleges or universities have to determine what credit in what courses to award for what AP or IB grades. The courses of interest to mathematical science departments are AP Calculus AB, AP Calculus BC, AP Statistics, and IB Mathematics.

The second type of courses is dual enrollment courses (also called concurrent enrollment courses). Credit for these courses is arranged by way of agreements between a high school and a college or university. Sometimes these courses are taught on a college campus with a mix of school and college students, and sometimes they are taught in a high school. The teacher may be either a college teacher or a high school teacher, and there may or may not be an examination fashioned by the college. The mathematics courses range from college algebra through the calculus sequence.

Both AP course enrollments and dual enrollment course enrollments have been increasing, and are likely to continue for the near future.

What do we know? There is considerable information about students who take AP Calculus, less about students who take AP Statistics, and IB programs are relatively few compared to AP. David Bressoud has written extensively about the interaction of calculus in high school and calculus in college.

What would we like to know? The literature on the effectiveness of dual enrollment courses paints a mixed picture of their effectiveness. There is a professional organization that promotes concurrent enrollment courses. The central question is this: Do dual enrollment courses increase learning in postsecondary education? If so, how? The same question can be asked about AP or IB courses. Studying college material in high school can hasten finishing college, but do they increase learning?

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Appendix

The six core competencies for quantitative reasoning (Boersma, et al., 2011)

1. *Interpretation*: Ability to glean and explain mathematical information presented in various forms (e.g. equations, graphs, diagrams, tables, words).
2. *Representation*: Ability to convert information from one mathematical form (e.g. equations, graphs, diagrams, tables, words) into another.
3. *Calculation*: Ability to perform arithmetical and mathematical calculations.
4. *Analysis/Synthesis*: Ability to make and draw conclusions based on quantitative analysis.
5. *Assumptions*: Ability to make and evaluate important assumptions in estimation, modeling, and data analysis.
6. *Communication*: Ability to explain thoughts and processes in terms of what evidence is used, how it is organized, presented, and contextualized.

The five strands of mathematical proficiency from Adding It Up (National Research Council, 2001)

1. *Conceptual understanding*: Comprehension of mathematical concepts, operations and relations.
2. *Procedural fluency*: Skill in carrying out procedures flexibly, accurately, efficiently, and appropriately.
3. *Strategic competence*: Ability to formulate, represent, and solve mathematical problems.
4. *Adaptive reasoning*: Capacity for logical thought, reflection, explanation, and justification.
5. *Productive disposition*: Habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence in one's own efficacy.

The mathematical practice standards from CCSSM (2010).

1. *Make sense of problems and persevere in solving them:* Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution.

2. *Reason abstractly and quantitatively***:** Mathematically proficient students make sense of quantities and their relationships in problem situations.

3. *Construct viable arguments and critique the reasoning of others:* Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments.

4. *Model with mathematics:* Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace.

5. *Use appropriate tools strategically:* Mathematically proficient students consider the available tools when solving a mathematical problem.

6. *Attend to precision:* Mathematically proficient students try to communicate precisely to others.

7. *Look for and make use of structure:* Mathematically proficient students look closely to discern a pattern or structure.

8. *Look for and express regularity in repeated reasoning:* Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts.

Four of the five elements of effective thinking (Burger and Starbird, 2012).

- Earth – Understand deeply. Don't face complex issues head-on; first understand simple ideas deeply. Clear the clutter and expose what is really important.
- Fire – Ignite insights by making mistakes. Fail to succeed. Intentionally get it wrong to inevitably get it more right. Mistakes are great teachers – they highlight unforeseen opportunities and holes in your thinking.
- Air – Raise questions. Constantly create questions to clarify and extend your understanding. What's the real question? Working on the wrong question can waste a lifetime. Be your own Socrates.
- Water – Follow the flow of ideas. Look back to see where ideas came from and then look ahead to see where the ideas may lead. A new idea is a beginning, not an end.

Conclusions from How People Learn (Bransford, Brown & Cocking, 2001).

1. Students come to the classroom with preconceptions about how the world works. If their initial understanding is not engaged, they may fail to grasp the new concepts and information that they are taught, or they may learn them for purposes of a test but revert to their preconceptions outside the classroom.
Therefore: Teachers must draw out and work with preexisting understandings that their students bring to them.

Comment [1]: Something's funny about the original numbering in this list. E.g., what does 1T mean? (I think 1T means something like "conclusion from 1"). I restructured the list a bit.

2. To develop competence in an area of inquiry, students must: (a) have a deep foundation of factual knowledge, (b) understand facts and ideas in the context of a conceptual framework, and (c) organize knowledge in ways that facilitate retrieval and application.

Therefore: Teachers must teach some subject matter in depth, providing many examples in which the same concept is at work and providing a firm foundation of factual knowledge. Burger and Starbird (2012) get at this in several ways. While giving advice on how to understand deeply, they say, "Sweat the small stuff." (p. 25). They note that when studying some complex issue, instead of attacking it in its entirety, find one small element of it and solve that part completely.

3. A “metacognitive” approach to instruction can help students learn to take control of their own learning by defining learning goals and monitoring their progress in achieving them. Burger and Starbird’s five elements are aimed at students (and others) taking control of their own learning. Although there are anecdotes from their classrooms that illustrate the five elements in action, the real message is to the learner-thinker.