Math 224, Fall 2007
Exam 2 Solutions

- You have 1 hour and 20 minutes.
- No notes, books, or other references.
- You are permitted to use Maple during this exam, but you must start with a blank worksheet. Start by typing with(linalg):
- YOU MUST SHOW ALL WORK TO RECEIVE CREDIT. ANSWERS FOR WHICH NO WORK IS SHOWN WILL RECEIVE NO CREDIT (UNLESS SPECIFICALLY STATED OTHERWISE).
- Good luck! Eat candy as necessary!

Name:

"On my honor, I have neither given nor received any aid on this examination."

Signature:

Question	Score	Maximum
1		8
2		20
3		10
4		10
5		20
6		6
7		6
8		10
9		10
Bonus		10
Total		100

1. (a) (5 points) Find the volume of the 3-box in $\mathbf{R}^4$ with *vertices* $(1,0,0,1)$, $(-1,2,0,1)$, $(3,0,1,1)$, and $(-1,4,0,1)$.
The 3-box is determined by the vectors

$$\begin{aligned}\mathbf{v_1} &= (-1,2,0,1)-(1,0,0,1) = [-2,2,0,0]\\ \mathbf{v_2} &= (3,0,1,1)-(1,0,0,1) = [2,0,1,0]\\ \mathbf{v_3} &= (-1,4,0,1)-(1,0,0,1) = [-2,4,0,0]\end{aligned}$$

The volume of the 3-box is given by

$$V = \sqrt{\det(A^T A)},$$

where

$$A = \begin{bmatrix} -2 & 2 & -2 \\ 2 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Using Maple, we compute

$$V = 4.$$

Note that even though A has a row of zeros, $A^T A$ does *not* have a row of zeros. Moreover, A is a 4×3 matrix, so $\det A$ is not defined.

(b) (3 points) Your friend (who, sadly, is not enrolled in Linear Algebra) claims that there is no such thing as 4-space, and thus, there is no such thing as a 3-box in $\mathbf{R}^4$. State the precise definition of a m-box in $\mathbf{R}^n$, where $m \leq n$, and explain to your friend why this definition makes sense (in terms of how we think of boxes in $\mathbf{R}^2$ and $\mathbf{R}^3$).
Let $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_m}$ be m independent vectors in $\mathbf{R}^n$ for $m \leq n$. The m-box in $\mathbf{R}^n$ determined by these vectors is the set of all vectors $\mathbf{x}$ satisfying

$$\mathbf{x} = t_1\mathbf{a_1} + t_2\mathbf{a_2} + \ldots t_m\mathbf{a_m}$$

for $0 \leq t_i \leq 1$, $i = 1, 2, \ldots, m$.

2. Let

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

(a) (2 points) Find the characteristic polynomial of A.
The characteristic polynomial is

$$p(\lambda) = (1-\lambda)(\lambda^2 + 4\lambda + 4) = (1 - \lambda(\lambda+2)^2.$$

(b) (2 points) Find the eigenvalues of A.
We find the eigenvalues of A by solving $p(\lambda) = 0$. We obtain

$$\lambda_1 = 1 \text{ and } \lambda_2 = \lambda_3 = -2.$$

(c) (4 points) Find the eigenvectors of A.
The eigenvectors corresponding to $\lambda_1 = 1$ are all vectors of the form

$$\mathbf{v_1} = \begin{bmatrix} r \\ -r \\ r \end{bmatrix},$$

where $r \neq 0$. The eigenvectors corresponding to $\lambda_2 = \lambda_3$ are all vectors of the form

$$\mathbf{v_2} = \begin{bmatrix} -r-s \\ r \\ s \end{bmatrix},$$

where r, s are not both equal to 0.

(d) (4 points) Explain why A must be diagonalizable.
The algebraic and geometric multiplicity of λ_1 are both equal to 1. The algebraic and geometric multiplicity of $\lambda_2 = \lambda_3 = -2$ are both equal to 2. Thus the algebraic multiplicity of each eigenvalue of A is equal to the geometric multiplicity, so A must be diagonalizable.

(e) (4 points) Find an invertible matrix C and a diagonal matrix D such that $C^{-1}AC = D$.
We construct the matrix C whose column vectors consist of independent eigenvectors of A:

$$C = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Then D is the diagonal matrix whose diagonal entries are the eigenvalues of A:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

(f) (4 points) Find A^k in terms of k.
We can rewrite $C^{-1}AC = D$ as $A = CDC^{-1}$. Thus $A^k = CD^kC^{-1}$. Since D is a diagonal matrix,

$$D^k = \begin{bmatrix} 1^k & 0 & 0 \\ 0 & (-2)^k & 0 \\ 0 & 0 & (-2)^k \end{bmatrix}.$$

Performing the matrix multiplication in Maple, we obtain

$$A^k = \begin{bmatrix} 1 & 1-(-2)^k & 1-(-2)^k \\ -1+(-2)^k & -1+2(-2)^k & -1+(-2)^k \\ 1-(-2)^k & 1-(-2)^k & 1 \end{bmatrix}.$$

3. Suppose that $\det A = 7$, where

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

Evaluate the following:

(a) (5 points) $\begin{vmatrix} 2a & 2b & 2c \\ 3d-a & 3e-b & 3f-c \\ 2g & 2h & 2i \end{vmatrix} = 2 \cdot 3 \cdot 2 \cdot \det A = 84$

(b) (5 points) $\begin{vmatrix} a+2d & b+2e & c+2f \\ 3g & 3h & 3i \\ d & e & f \end{vmatrix} = -1 \cdot 3 \cdot \det A = -21$

4. (a) (4 points) What are the possible values of of the determinant of an $n \times n$ matrix A such that $AA^T = I$?
If $AA^T = I$, then

$$\begin{aligned} \det(AA^T) &= \det(I) \\ \det(A)\det(A^T) &= 1 \\ \det(A)\det(A) &= 1 \\ (\det(A))^2 &= 1 \\ \det(A) &= \pm 1 \end{aligned}$$

(b) (4 points) Let A be an $n \times n$ invertible matrix. Prove that

$$\det\left(A^{-1}\right) = \frac{1}{\det A}.$$

Since A is invertible, $A^{-1}A = I$. Thus:

$$\begin{aligned} \det(A^{-1}A) &= \det(I) \\ \det(A^{-1})\det A &= 1 \\ \det(A^{-1}) &= \frac{1}{\det A} \end{aligned}$$

5. (a) (5 points) Let A be an $n \times n$ matrix such that A^k is equal to the zero matrix for some positive integer k. Show that the only eigenvalue of A is 0.

Suppose that λ is an eigenvalue of A with corresponding eigenvector $\mathbf{v}$. Note that since $A\mathbf{v} = \lambda\mathbf{v}$, $A^k\mathbf{v} = \lambda^k\mathbf{v}$. Since $A^k = 0$, $\lambda^k\mathbf{v} = 0$. Since $\mathbf{v}$ is a non-zero vector (by definition of eigenvector), we conclude that $\lambda^k = 0$. Thus $\lambda = 0$.

(b) (5 points) Let λ be an eigenvalue of an invertible matrix A. Show that $\lambda \neq 0$ and that λ^{-1} is an eigenvalue of A^{-1}.

First, note that if $\lambda = 0$, then $\det(A - 0I) = \det A = 0$, so A is not invertible. Since A is invertible, $\lambda \neq 0$. Next, we show that $A^{-1}\mathbf{v} = \lambda^{-1}\mathbf{v}$:

$$\begin{aligned} A\mathbf{v} &= \lambda\mathbf{v} \\ A^{-1}A\mathbf{v} &= \mathbf{v} \\ A^{-1}\lambda\mathbf{v} &= \mathbf{v} \\ A^{-1}\mathbf{v} &= \frac{1}{\lambda}\mathbf{v} \end{aligned}$$

Thus λ^{-1} is an eigenvalue of A^{-1}.

(c) (5 points) Suppose that A and B are two $n \times n$ matrices. Show that if A is similar to B, then A^2 is similar to B^2.

Since A is similar to B, there is an invertible matrix C such that $A = C^{-a}BC$. Squaring both sides, we obtain:

$$\begin{aligned} A^2 &= (C^{-1}BC)^2 \\ A^2 &= C^{-1}BCC^{-1}BC \\ A^2 &= C^{-1}B^2C \end{aligned}$$

Thus A^2 is similar to B^2.

(d) (5 points) Suppose that A is a diagonalizable $n \times n$ matrix and has only 1 and -1 as eigenvalues. Show that $A^2 = I_n$, where I_n is the $n \times n$ identity matrix.

Since A is diagonalizable, there is an invertible matrix C such that $C^{-1}AC = D$, where D is a diagonal matrix whose diagonal entries are all ± 1 (the eigenvalues of A). Note that A and D are not necessarily 2×2 matrices, as the multiplicities of the eigenvalues could be greater than 1. Since the diagonal entries of D are all ± 1, $D^2 = I_n$. Then:

$$\begin{aligned} A^2 &= (CDC^{-1})^2 \\ &= CD^2C^{-1} \\ &= CI_nC^{-1} \\ &= I_n \end{aligned}$$

6. Suppose that A is a diagonalizable matrix with characteristic polynomial

$$p(\lambda) = (\lambda - 1)(\lambda - 3)^2(\lambda - 4)^3.$$

(a) (2 points) Find the size of the matrix A.
A is a 6×6 matrix.

(b) (4 points) Find the dimension of E_4, the eigenspace corresponding to the eigenvalue 4.
Since A is diagonalizable, the algebraic multiplicity of each eigenvalue is equal to its geometric multiplicity. Since the algebraic multiplicity of $\lambda = 4$ is 3, the geometric multiplicity is also 3. Thus the dimension of E_4 is 3.

7. (6 points) Suppose that A is a diagonalizable matrix with characteristic polynomial

$$p(\lambda) = \lambda^2(\lambda - 3)(\lambda + 2)^3.$$

Find the dimension of the nullspace of A.

The nullspace of A is the set of vectors $\mathbf{v}$ that satisfy $A\mathbf{v} = \mathbf{0}$. Note that $A\mathbf{v} = \mathbf{0}$ is equivalent to $(A - 0I)\mathbf{v} = \mathbf{0}$. Thus the nullspace of A is the eigenspace corresponding to the $\lambda = 0$ eigenvalue. Since $\lambda = 0$ is an eigenvalue of algebraic multiplicity 2, and A is diagonalizable, the nullspace of A has dimension 2.

8. (10 points) Under what conditions does the 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

have no real eigenvalues?

A has no real eigenvalues if and only if $p(\lambda) = \det(A - \lambda I) = 0$ has no real roots.

$$\begin{aligned} p(\lambda) &= (a - \lambda)(d - \lambda) - bc \\ &= \lambda^2 - (a + d)\lambda + ad - bc \end{aligned}$$

Using the quadratic formula to solve $p(\lambda) = 0$, we obtain

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2a}.$$

Thus A has no real eigenvalues if and only if

$$(a-d)^2 - 4(ad-bc) < 0.$$

9. (10 points) Classify each of the following statements as True or False. No explanation is necessary.

(a) Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear transformation with standard matrix representation A. The image under $T \circ T$ of an n-box in $\mathbf{R}^n$ of volume V is a box in $\mathbf{R}^n$ of volume $\det(A^2) \cdot V$.
True.

(b) Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear transformation with standard matrix representation A. The image under $T \circ T \circ T$ of an n-box in $\mathbf{R}^n$ of volume V is a box in $\mathbf{R}^n$ of volume $\det(A^3) \cdot V$.
False. The volume-change factor is $|\det(A^3)|$. We need the absolute value here since $\det(A^3) = \det(A)^3$ might be negative.

(c) If $\mathbf{v}$ is an eigenvector of an invertible matrix A, then $c\mathbf{v}$ is an eigenvector of A^{-1} for all non-zero scalars c.
True. If $\mathbf{v}$ is an eigenvector of A with corresponding eigenvalue λ, then we have seen previously that $A^{-1}\mathbf{v} = \lambda^{-1}\mathbf{v}$. Then $A^{-1}c\mathbf{v} = \lambda^{-1}c\mathbf{v}$, so $c\mathbf{v}$ is an eigenvector of A^{-1} for all non-zero scalars c.

(d) If λ is an eigenvalue of a matrix A, then λ is an eigenvalue of $A + cI$ for all scalars c.
False. We have seen that $\lambda + c$ is an eigenvalue of $A + cI$.

(e) If A is a 3×3 matrix with characteristic polynomial $p(\lambda) = (\lambda - 2)^2(\lambda - 3)$, then there is at most one non-zero vector $\mathbf{v}$ such that $A\mathbf{v} = 3\mathbf{v}$.
False.

(f) If $\mathbf{R}^n$ has a basis consisting of eigenvectors of an $n \times n$ matrix A, then A is diagonalizable.
True.

(g) If A and B are similar matrices, then $\det A = \det B$.
True.

(h) If an $n \times n$ matrix A is diagonalizable, then there is a unique diagonal matrix D that is similar to A.
False. D is not unique.

(i) If A is any $n \times n$ matrix, then the determinant of A is equal to the product of the diagonal entries in A.
False. If A is not a diagonal matrix, then the determinant of A is no necessarily equal to the product of its diagonal entries.

(j) If an $n \times n$ matrix A does not have n distinct eigenvalues, then A is not diagonalizable.
False.

Bonus (10 points). Recall that the Fibonacci sequence is given by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Find a closed-form formula for F_n in terms of n.